Discrete Mathematics

Discrete Probability

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7.3: Bayes's Theorem

Bayes's theorem

Example

We have two boxes. The first contains two green balls and seven red balls; the second contains four green balls and three red balls. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?

Bayes's theorem

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Suppose that E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$. Then:

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}.$$

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Example

Suppose that one person in 100,000 has a particular rare disease for which there is a diagnostic test. This test is correct 99.0% of the time when given to a person selected at random who hast he disease; it is correct 99.5% of the time when given to a person selected at random who does not have the disease. Find the probabilities that a person who tests positive has the disease and that a person who tests negative does not have the disease.

Bayes's theorem

Generalized Bayes's Theorem

Suppose that E is an event from a sample space S such that F_1, F_2, ..., F_n are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$. Assume that $p(E) \neq 0$ and $p(F_i) \neq 0$ for all i. Then:

$$p(F_j \mid E) = \frac{p(E \mid F_j)p(F_j)}{\sum_{i=1}^{n} p(E \mid F_i)p(F_i)}.$$

7.4: Expected Value and Variance

Expected values

Definition

The *expected value*, also called the *expectation* or *mean*, of the random variable X on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)X(s).$$

The *deviation* of X as $s \in S$ is $X(s) - E(X)$, the (signed) distance from the mean.

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Example

A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

Expected values

Theorem

If X is a random variable and $p(X = r)$ is the probability that $X = r$, so that $p(X = r) = \sum_{s \in S, X(s) = r} p(s)$, then

$$E(X) = \sum_{s \in X(S)} p(X = r) \cdot r.$$

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What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

Theorem

The expected number of successes when n mutually independent Bernoulli trials are performed, where p is the probability of success on each trial, is np.

Linearity of expectations

Theorem

If X_i, $i = 1, 2, \ldots, n$, are random variables on S and if a and b are real numbers, then

1. $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$.
2. $E(aX + b) = aE(X) + b$.

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Example

A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the expected number of hats returned correctly?

Linearity of expectations

Example

The ordered pair (i, j) is called an *inversion* in a permutation of the first n positive integers if $i < j$ but j preceeds i in the permutation. (For instance, $(1, 5)$ is an inversion in the permutation $(3, 5, 1, 4, 2)$, but $(1, 2)$ is not.)

Lineraity of expectations

Example

Let $I_{i,j}$ be the random variable on the set of permutations which is 1 is (i,j) is an inversion and 0 if not. If X is the random variable whose value is the total number of permutations, it follows that

$$X = \sum_{1 \le i < j \le n} I_{i,j}.$$

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Note that it is equally likely in a randomly chosen permutation for i to precede j as it is for j to precede i. Hence, $E(I_{i,j}) = 1/2$.

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Note that it is equally likely in a randomly chosen permutation for i to precede j as it is for j to precede i. Hence, $E(I_{i,j}) = 1/2$. Because there are $\binom{n}{2}$ pairs i and j with $1 \leq i < j \leq n$ and by linearity of expectation, we have

$$E(X) = \sum_{1 \leq i < j \leq n} E(I_{i,j}) = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}.$$

The geometric distribution

Definition

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Theorem

If the random variable X is geometrically distributed with parameter p, then $E(X) = 1/p$.

Independent random variables

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Theorem

If X and Y are independent random variables on a sample space S, then $E(XY) = E(X)E(Y)$.

Variance

Definition

Let X be a random variable on a sample space S. The *variance* of X, denoted by $V(X)$, is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

That is, $V(X)$ is the weighted average of the square of the deviation of X.

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The *standard deviation* of X, denoted $\sigma(X)$, is defined to be $\sqrt{V(X)}$.

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If X is a random variable on a sample space S and $E(X) = \mu$, then $V(X) = E((X - \mu)^2)$.

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If X is a random variable on a sample space S and $E(X) = \mu$, then $V(X) = E((X - \mu)^2)$.

Example

What is the variance of the random variable X, when X is the number that comes up when a fair die is rolled?

Variance

Theorem (Bienaymé's Formula)

If X and Y are two independent random variables on a sample space S, then $V(X+Y) = V(X) + V(Y)$. Furthermore, if X_i, $i = 1, \ldots, n$, are pairwise independent random variables on S, then

$$V(X_1 + \cdots + X_n) = V(X_1) + \cdots + V(X_n).$$

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Example

What is the variance of the number of successes when n independent Bernoulli trials are performed, where on each trial p is the probability of success and q is the probability of failure?

Chebyshev's inequality

Theorem (Chebyshev's Inequality)

Let X be a random variable on a sample space S with probability function p. If r is a positive real number, then

$$p(|X(s) - E(X)| \geq r) \leq V(X)/r^2.$$

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Suppose that X is the random variable that counts the number of tails when a fair coin is tossed n times. (Note that X carries a Bernoulli distribution with parameter $p = 1/2$.) It follows that $E(X) = n/2$ and $V(X) = n/4$. Applying Chebyshev's inequality with $r = \sqrt{n}$ shows that

$$p(|X(s) - n/2| \geq \sqrt{n}) \leq (n/4)/(\sqrt{n})^2 = 1/4.$$