Discrete Mathematics

Counting

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6.4: Binomial Coefficients and Identities

The Binomial Theorem

Theorem

Let x and y be variables, and let n be a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$
$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

The Binomial Theorem

Corollary

Let $n \geq 0$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

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Let $n \geq 0$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Similarly, when $n > 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Pascal's identity and triangle

Theorem (Pascal's identity)

Let n and k be positive integers with $n \geq k$. Then:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Other identities involving binomial coefficients

Theorem (Vandermonde's identity)

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Other identities involving binomial coefficients

Theorem

If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Other identities involving binomial coefficients

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Theorem

Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

6.5: Generalized Permutations and Combinations

Counting with repetition

Theorem

The number of r-permutations of a set of n objects with repetition allowed is n^r.

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Theorem

There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

Counting with repetition

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Counting with repetition

Summary

Type	Repetition allowed?	Formula
r-permutations	No	$\frac{n!}{(n-r)!}$,
r-combinations	No	$\frac{n!}{r!(n-r)!}$,
r-permutations	Yes	n^r,
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$.

Permutations with indistinguishable objects

Theorem

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is given by

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Distributing objects into boxes

Example (Distinguishable objects and distinguishable boxes)

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

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Theorem

The number of ways to distribute n distinguishable objects into k distinguishable boxes such that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

Distributing objects into boxes

Example (Indistinguishable objects and distinguishable boxes)

How many ways are there to place 10 indistinguishable balls into eight distinguishable boxes?

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Theorem

There are $C(n+r-1, n-1)$ ways to place r indistinguishable objects into n distinguishable boxes.

Distributing objects into boxes

Remark (Distinguishable objects and indistinguishable boxes)

There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes. Let $S(n,j)$, called *Stirling numbers of the second kind*, denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. One can show that

$$S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Consequently, the number of ways to distribute n distinguishable objects into k indistinguishable boxes equals

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k}\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Distributing objects into boxes

Remark (Indistinguishable objects and indistinguishable boxes)

Observe that distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in nondecreasing order. If $a_1 + \cdots + a_j = n$, where $a_1 \geq \ldots \geq a_j$ are positive integers, we say $a_1, \ldots, a_j$ is a *partition* of n into j parts. We see that if $p_k(n)$ is the number of partitions of n into at most k parts, then there are $p_k(n)$ ways to distribute n indistinguishable objects into k indistinguishable boxes. No simple closed form exists for this number.