Discrete Mathematics

Induction and Recursion

Prof. Steven Evans

5.1: Mathematical Induction

Mathematical induction

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

1. Basis step: We verify that $P(1)$ is true.
2. Inductive step: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Mathematical induction

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption $P(k+1)$ must also be true. The assumption is called the *inductive hypothesis*.

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Definition

Expressed as a rule of inference, the proof technique of *induction* can be stated as

$$\frac{P(1) \qquad \forall k(P(k) \rightarrow P(k+1))}{\therefore \forall n P(n)}.$$

Mathematical induction

Why is this a valid proof technique?

Natural numbers are *well-ordered*: every nonempty subset of $\mathbb{N}$ has a least element. So, suppose we know that $P(1)$ and that $P(k) \rightarrow P(k+1)$ are true. To show $P(n)$ must be true for all n, suppose otherwise, that there is some n for which it is false. Take m to be the smallest such value for which $P(m)$ is false. The value m cannot be 1, since $P(1)$ is true, and hence $m-1$ must be a positive integer. But then the truth of $P(m-1) \rightarrow P(m)$ contradicts $\neg P(m)$.

Examples of mathematical induction

Example

Use mathematical induction to prove this formula for the sum of a finite number of terms fo a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1},$$

where $r \neq 1$ and n is a nonnegative integer.

Examples of mathematical induction

Example

The *harmonic numbers* H_j are defined by

$$H_j = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{j}.$$

Use mathematical induction to show that $H_{2^n} \geq 1 + \frac{n}{2}$, whenever n is a nonnegative integer.

Examples of mathematical induction

Example

Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

5.2: Strong Induction and Well-Ordering

Strong induction

In a proof by mathematical induction, the inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ is also true. In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers not exceeding k, then $P(k+1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$.

Strong induction

In a proof by mathematical induction, the inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ is also true. In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers not exceeding k, then $P(k+1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$.

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

1. Basis step: We verify that the proposition $P(1)$ is true.
2. Inductive step: We show that the conditional statement $(P(1) \wedge \cdots \wedge P(k)) \to P(k+1)$ is true for all positive k.

Examples of strong induction

Example

Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

The well-ordering property

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Example

In a round-robin tournament, every player plays every other player exactly once and each match has a winner and a loser. We say that the players $p_1, \ldots, p_m$ form a *cycle* if p_1 beats $p_2, \ldots, p_{m-1}$ beats p_m, and p_m beats p_1. Use the well-ordering principle to show that if there is a cycle of length m ($m \geq 3$) among the players in a round-robin tournament, there must be a cycle amongst just three of these players as well.

5.3: Recursive Definitions and Structural Induction

Recursively defined functions

Definition

We use two steps to define a function with the set of nonnegative integers as its domain:

1. Basis step: Specify the value of the function at zero.
2. Recursive step: Give a rule for finding its value at an integer from its values at smaller integers.

Recursively defined functions

Example

Give a recursive definition of a^n, where a is a nonzero real number and n is a nonnegative integer.

Trees

Definition

A tree is a special type of *graph*; a graph is made up of vertices and edges connecting some pairs of vertices.

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A tree is a special type of *graph*; a graph is made up of vertices and edges connecting some pairs of vertices.
Rooted trees, which have a distinguished vertex called the *root*, can be recursively defined as follows:

1. Basis step: A single vertex r is a rooted tree.
2. Suppose that $T_1, \ldots, T_n$ are disjoint rooted trees with roots $r_1, \ldots, r_n$ respectively. Then the graph formed by starting with a root r, which is not in any of the rooted trees $T_1, \ldots, T_n$, and adding an edge from r to each of the vertices $r_1, \ldots, r_n$ is also a rooted tree.

Trees

Definition

Extended binary trees can be defined recursively by these steps:

1. Basis step: The empty set is an extended binary tree.
2. Recursive step: If T_1 and T_2 are disjoint extended binary trees, there is an extended binary tree denoted by $T_1 \cdot T_2$ consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2 (when these trees are nonempty).

Trees

Definition

Full binary trees can be defined recursively by these steps:

1. Basis step: There is a full binary tree consisting only of a single vertex r.
2. Recursive step: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree $T_1 \cdot T_2$ consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and right subtree T_2.

Trees

Definition

We define the *height* $h(T)$ of a full binary tree T recursively:

1. Basis step: The height of the full binary tree T consisting of only a root r is $h(T) = 0$.
2. Recursive step: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has height $h(T) = 1 + \max(h(T_1), h(T_2))$.

Trees

Remark

Letting $n(T)$ denote the number of vertices in a full binary tree, we observe that $n(T)$ satisfies the following recursive formula:

1. Basis step: The number of vertices $n(T)$ of the full binary tree T consisting of only a root r is $n(T) = 1$.
2. Recursive step: If T_1 and T_2 are full binary trees, then the number of vertices of the full binary tree $T = T_1 \cdot T_2$ is $n(T) = 1 + n(T_1) + n(T_2)$.

Trees

Remark

Letting $n(T)$ denote the number of vertices in a full binary tree, we observe that $n(T)$ satisfies the following recursive formula:

1. Basis step: The number of vertices $n(T)$ of the full binary tree T consisting of only a root r is $n(T) = 1$.
2. Recursive step: If T_1 and T_2 are full binary trees, then the number of vertices of the full binary tree $T = T_1 \cdot T_2$ is $n(T) = 1 + n(T_1) + n(T_2)$.

Theorem

If T is a full binary tree, then $n(T) \leq 2^{h(T)+1} - 1$.