MATH 1A	Fall 2012
2009 Midterm 2 : 10/26/2012	
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1. Have $\frac{-1}{x}\sin(\ln x)e^{\cos(\ln x)}$ by chain rule.

2. Since $y'(x) = \frac{1}{x^2}(1 - \ln x)$, slope of tangent line at 1 is 1. Therefore $y = x - 1$.

3. Taking derivatives of both and solving for y' obtain $y' = \dfrac{ye^{x/y} - y}{xe^{x/y} - y^2}$. Noting $e^{x/y} = x - y$ this can be rewritten as $y' = \dfrac{xy - 2y^2}{x^2 - xy - y^2}$.

4. Have $y(t) = y(0)e^{-.0005t} = e^{-.0005t}$.

5. Set $f(x) = \sqrt{x}$. Have $f(x) \approx f(4) + f'(4)(x - 4)$ for x near 4. Therefore

$$f(4.1) \approx 2 - \frac{1}{4}\frac{1}{10} = \frac{79}{40}.$$

6. ① Note $f(-3) = 9$ and $f(5) = 65$. ② Since $f'(x) = 3x^2 - 12$, have $f'(c) = 0$ if and only if $c = \pm 2$. Note $f(2) = -16$ and $f(-2) = 16$. ③ Comparing the values conclude absolute max is 65 and absolute min is -16.

7. Set $f(x) = e^x + x$. ① If $f(x)$ had more than one root, then $f(x)$ would have at least two. Since $f(x)$ differentiable on all numbers, Rolle's theorem implies there exists a point c where $0 = f'(c) = e^c + 1$. However $e^x + 1 > 1 > 0$ for all numbers x. Conclude $f(x)$ has at most one root. ② Note $f(1) > 0$ and $f(-1) < 0$. Since $f(x)$ continuous on all numbers, I.V.T implies that $f(x)$ has a root.

8. By L'Hospital's rule have

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\\ &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}\\ &= \lim_{x\to 0}\frac{4\sec^2 x\tan^2 x + 2\sec^4 x}{6} = 1/3.\end{aligned}$$

9. Note $y = \sqrt[3]{x(x-1)(x+1)}$. Domain $\mathbb{R}$. Zeros at $x = 0, \pm 1$. $\lim_{x\to+\infty} y(x) = +\infty$ and $\lim_{x\to-\infty} y(x) = -\infty$. Since $y'(x) = \frac{1}{3}(x^3 - x)^{-2/3}(3x^2 - 1)$, have critical points $x =$

$0, \pm 1, \pm \frac{1}{\sqrt{3}}$ with

$$y' > 0 \text{ for } |x| > \frac{1}{\sqrt{3}} \text{ and } y' < 0 \text{ for } |x| < \frac{1}{\sqrt{3}}.$$

Local max at $\frac{-1}{\sqrt{3}}$ and local min at $\frac{1}{\sqrt{3}}$. Vertical tangent at $0, \pm 1$. Obtain graph plotted here:

`http://www.graphsketch.com/?eqn1_color=1&eqn1_eqn=root%28x^3-x%2C3%29&eqn2_color=2&eqn2_eqn=&eqn3_color=3&eqn3_eqn=&eqn4_color=4&eqn4_eqn=&eqn5_color=5&eqn5_eqn=&eqn6_color=6&eqn6_eqn=&x_min=-1.5&x_max=1.5&y_min=-1&y_max=1&x_tick=.5&y_tick=.5&x_label_freq=1&y_label_freq=1&do_grid=0&do_grid=1&bold_labeled_lines=0&bold_labeled_lines=1&line_width=4&image_w=850&image_h=525`

10. Set $g(x) = \dfrac{\ln x}{x}$. Domain $x > 0$. Zero at $x = 1$. $\lim_{x \to +\infty} g(x) = 0$ and $\lim_{x \to 0^+} g(x) = -\infty$. Since $g'(x) = \frac{1}{x^2}(1 - \ln x)$ have critical point at $x = e$ with $g'(x) > 0$ for $x < e$ and $g'(x) < 0$ for $x > 3$. Note local max at $x = e$. Obtain graph plotted here:

`http://www.graphsketch.com/?eqn1_color=1&eqn1_eqn=ln%28x%29%2Fx&eqn2_color=2&eqn2_eqn=&eqn3_color=3&eqn3_eqn=&eqn4_color=4&eqn4_eqn=&eqn5_color=5&eqn5_eqn=&eqn6_color=6&eqn6_eqn=&x_min=0&x_max=4&y_min=-10&y_max=10&x_tick=1&y_tick=1&x_label_freq=1&y_label_freq=1&do_grid=0&do_grid=1&bold_labeled_lines=0&bold_labeled_lines=1&line_width=4&image_w=850&image_h=525`

Set $f(x) = e^{g(x)}$. Domain $x > 0$. Never zero. $\lim_{x \to +\infty} f(x) = 1$ and $\lim_{x \to 0^+} f(x) = 0$. Note $f'(x) = g'(x)f(x)$ by chain rule where $f(x) > 0$ for all numbers x. Have critical point at $x = e$ with $f'(x) > 0$ for $x < e$ and $f'(x) < 0$ for $x > 3$. Local max at $x = e$. Obtain graph plotted here:

`http://www.graphsketch.com/?eqn1_color=1&eqn1_eqn=exp%28ln%28x%29%2Fx%29&eqn2_color=2&eqn2_eqn=&eqn3_color=3&eqn3_eqn=&eqn4_color=4&eqn4_eqn=&eqn5_color=5&eqn5_eqn=&eqn6_color=6&eqn6_eqn=&x_min=0&x_max=20&y_min=0&y_max=2&x_tick=.5&y_tick=.5&x_label_freq=2&y_label_freq=1&do_grid=0&do_grid=1&bold_labeled_lines=0&bold_labeled_lines=1&line_width=4&image_w=850&image_h=525`