Math 1A 2005 Midterm 2

1). The graph of f has a horizontal tangent precisely when $f'(x) = 0$. Since $f'(x) = 1-2\cos(x)$, this happens when $1 - 2\cos(x) = 0$, i.e. $\cos(x) = 1/2$. The values of x which satisfy this are $x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$, for $n \in \mathbb{Z}$.

2). We have $y' = 10(1 + 3x)^9(3)$ by the Chain Rule, so $y'(0) = 10(1 + 3 \cdot 0)^9(3) = 30$. The equation of the tangent line is $y - 1 = 30(x - 0)$, or $y = 30x + 1$.

3). Taking derivatives implicitly yields

$$\begin{aligned}
\frac{1}{2\sqrt{xy}} \cdot \frac{d}{dx}[xy] &= \frac{d}{dx}[x^2y] \\
\frac{1}{2\sqrt{xy}}(y + x\frac{dy}{dx}) &= 2xy + x^2\frac{dy}{dx} \\
y + x\frac{dy}{dx} &= 2\sqrt{xy}(2xy + x^2\frac{dy}{dx}) \\
(x - 2x^2\sqrt{xy})\frac{dy}{dx} &= 4(xy)^{3/2} - y \\
\frac{dy}{dx} &= \frac{4(xy)^{3/2} - y}{x - 2x^2\sqrt{xy}}
\end{aligned}$$

4). Notice $y = \frac{x}{2x-1} = \frac{1}{2}\left(\frac{2x}{2x-1}\right) = \frac{1}{2}\left(\frac{2x-1+1}{2x-1}\right) = \frac{1}{2}\left(1 + \frac{1}{2x-1}\right)$. Therefore $y' = -(2x-1)^{-2}, y'' = 4(2x-1)^{-3}, y''' = -24(2x-1)^{-4}$.

5). By the Chain Rule,

$$\begin{aligned}
\frac{d}{dx}[\ln(\ln(\ln(\ln(x))))] &= \frac{1}{\ln(\ln(\ln(x)))}\frac{d}{dx}[\ln(\ln(\ln(x)))] \\
&= \frac{1}{\ln(\ln(\ln(x)))}\frac{1}{\ln(\ln(x))}\frac{d}{dx}[\ln(\ln(x))] \\
&= \frac{1}{\ln(\ln(x)) \cdot \ln(\ln(\ln(x)))}\frac{1}{\ln x}\frac{d}{dx}[\ln x] \\
&= \frac{1}{x \cdot \ln x \cdot \ln(\ln(x)) \cdot \ln(\ln(\ln(x)))}
\end{aligned}$$

6). By the Product Rule, $\frac{d}{dx}[\sinh(x)\tanh(x)] = \frac{d}{dx}[\sinh(x)]\tanh(x) + \sinh(x)\frac{d}{dx}[\tanh(x)] = \cosh(x)\tanh(x) + \sinh(x)\operatorname{sech}^2(x) = \sinh(x)(1 + \operatorname{sech}^2(x))$.

7). Let $f(x) = \sqrt{x}$, and $a = 100$. Then $f(a) = 10$, and $f'(a) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$, so the linear approximation to f at a is $L(x) = f(a) + f'(a)(x - a) = 10 + \frac{1}{20}(x - 100)$. Since $99.8 \approx 100$, $\sqrt{99.8} = f(99.8) \approx L(99.8) = 10 + \frac{1}{20}(99.8 - 100) = 10 + \frac{1}{20}(-0.2) = 10 - 0.01 = 9.99$.

Alternative approach (with differentials): For $f(x)$ as above, we have $dy = f'(x)dx = \frac{dx}{2\sqrt{x}}$. For $a = 100, x = 99.8$, we have $dx = \Delta x = -0.2$, so $\sqrt{99.8} = \sqrt{100} + \Delta y \approx 10 + dy = 10 + \frac{-0.2}{2\sqrt{100}} = 9.99$.

8). We first find the critical numbers of f. Since $f'(x) = 3x^2 - 3$, $f'(x) = 0$ when $x = 1$ or $x = -1$. As we only consider values in $[0, 3]$, the only critical number we check is $x = 1$. Evaluating at the critical number and the endpoints, we find $f(0) = 1, f(1) = -1, f(3) = 19$, so the absolute minimum is -1 and the absolute maximum is 19.

9). We compute $f'(x) = \frac{1}{3x^{2/3}} - \frac{2}{3x^{1/3}}$. f' is undefined for $x = 0$, and is 0 when $\frac{1}{3x^{2/3}} = \frac{2}{3x^{1/3}}$ iff $x^{1/3} = 2x^{2/3}$ iff $x = 8x^2$ iff $x = 0, 1/8$. The critical numbers are thus $0, \frac{1}{8}$.

10). As f is a polynomial, it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also $f(0) = 1 = f(4)$, so f satisfies the hypotheses of Rolle's Theorem on $[0, 4]$. The conclusion is then that there exists at least one value c in $(0, 4)$ with $f'(c) = 0$. We have $f'(x) = 2x - 4$, which is 0 precisely when $c = 2$.

11). $f'(x) = 2xe^x + e^x x^2 = xe^x(2 + x)$, so $f' = 0$ when $x = 0, -2$. We see that $f'(x) < 0$ for $-2 < x < 0$ (e.g., substitute $x = -1$), and $f'(x) > 0$ when $x > 0$ or $x < -2$. So by the First Derivative Test, $(-2, 4/e^2)$ is a local maximum and $(0, 0)$ is a local minimum for f, and f is increasing on $(-\infty, -2) \cup (0, \infty)$ and decreasing on $(-2, 0)$.

12). Since $e^x - 1 - x\big|_{x=0} = 0 = x^2\big|_{x=0}$, we may use L'Hospital's Rule (0/0 indeterminate form) to evaluate the limit. We have $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \dfrac{e^x - 1}{2x}$ if the latter limit exists. Again, since $e^x - 1\big|_{x=0} = 0 = 2x\big|_{x=0}$, we apply L'Hospital's Rule again to conclude that $\lim_{x\to 0} \dfrac{e^x - 1}{2x} = \lim_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$. Thus the original limit is $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2} = \dfrac{1}{2}$.

13). Notice $\sin(x), \sinh(x)$ are continuous functions on $\mathbb{R}$, and $\sin(0) = 0 = \sinh(0)$. Thus we substitute $x = 0$ to obtain $\lim_{x\to 0} \dfrac{\sin(x)}{\sinh(x) + 1} = \dfrac{0}{0 + 1} = 0$.

14). Domain: f is undefined when $1 + \cos(x) = 0$, which occurs when $x = (2n + 1)\pi$, for $n \in \mathbb{Z}$. Thus the domain of f is $\{x \in \mathbb{R} \mid x \neq (2n + 1)\pi, n \in \mathbb{Z}\}$.
Local Extrema: $f'(x) = \dfrac{(1 + \cos(x))(\cos(x)) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} = \dfrac{\cos(x) + 1}{(1 + \cos(x))^2} = \dfrac{1}{1 + \cos(x)}$.
This is always > 0, and is undefined when $1 + \cos(x) = 0$, precisely where f is undefined. Thus f is always increasing, and has no local maxima or minima.
Behavior at infinity: Both $\cos(x), \sin(x)$ are periodic of period 2π, so f is also periodic with the same period. Also, $\sin(x)$ is odd and $1 + \cos(x)$ is even, so f is odd. Thus the graph of f is just obtained by horizontal translates of its restriction to $[-\pi, \pi]$. f also has vertical asymptotes at $x = (2n + 1)\pi$, $n \in \mathbb{Z}$.
Zeros: f has zeros where it is defined and $\sin(x) = 0$, i.e. when $x = 2n\pi$, $n \in \mathbb{Z}$.
Behavior at 0: As seen above, f has a root at 0, and is continuous at (and increasing in a neighborhood of) 0.

15). Domain: $\{x \in \mathbb{R} \mid x > 0\}$ (we only look at $x > 0$)
Local Extrema: $f(x) = e^{\frac{\ln(x)}{x}} \Rightarrow f'(x) = e^{\frac{\ln(x)}{x}} \left(\dfrac{1 - \ln(x)}{x^2}\right) = \dfrac{x^{1/x}(1 - \ln(x))}{x^2}$. Thus $f' = 0$ when $x = e$. For $0 < x < e$, $f'(x) > 0$, and for $x > e$, $f'(x) < 0$. Thus f has a local max at $(e, e^{1/e})$, is increasing on $(0, e)$, and decreasing on (e, ∞).
Zeros: $f(x) = e^{\frac{\ln(x)}{x}}$ is never 0 for $x > 0$.
Behavior at ∞: $\lim_{x\to\infty} \dfrac{\ln(x)}{x} = 0$, so $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} = e^0 = 1$.
Behavior at 0: Substituting 0 for x gives the non-indeterminate form $0^\infty = 0$, so $f(0) = 0$.