Math 1A
Midterm 2
2006

1. Differentiate $\sin(\cos(\tan(x)))$.

   Solution. If $f(x) = \sin(\cos(\tan(x)))$ then by the chain rule we have that

   $$f'(x) = [\cos(\cos(\tan(x)))] \cdot [-\sin(\tan(x)] \cdot [\sec^2(x)]$$

   The brackets are only there to emphasize the applications of the chain rule. $\square$

2. Find an equation of the tangent line to the curve $y = \frac{1}{\sin(x)+\cos(x)}$ at the point $(0, 1)$.

   Solution. $y' = \frac{-(\cos(x)-\sin(x))}{(\sin(x)+\cos(x))^2}$ so $y'(0) = \frac{-(1-0)}{(0+1)^2} = \frac{-1}{1} = -1$ and so the equation of the tangent line at the point $(0, 1)$ is

   $$y - 1 = -x$$

   $\square$

3. Find $\frac{dy}{dx}$ by implicit differentiation if $1 + x = \sin(xy^2)$.

   Solution. Differentiate both sides of $1 + x = \sin(xy^2)$ with respect to x:

   $$\begin{aligned} \frac{d}{dx}(1 + x) &= \frac{d}{dx}(\sin(xy^2)) \\ 1 &= \cos(xy^2)(y^2 + 2xyy') \\ \sec(xy^2) &= y^2 + 2xyy' \\ \sec(xy^2) - y^2 &= 2xyy' \\ y' &= \frac{\sec(xy^2) - y^2}{2xy} \end{aligned}$$

   $\square$

4. Find a formula for the nth derivative of x^{-3}.

Solution. The strategy here is to find the derivative for a few values of n (e.g. $n = 1, 2, 3$) and recognize a pattern. To this end, let $f(x) = x^{-3}$. Then

$$\begin{aligned} f'(x) &= -3x^{-4} \\ f''(x) &= 12x^{-5} \\ f'''(x) &= -60x^{-6} \end{aligned}$$

We see that the exponent keeps coming down and the sign keeps alternating. Formally,

$$f^{(n)}(x) = (-1)^n \frac{(n+2)!}{2} x^{-3-n}$$

□

5. Differentiate $x^{\sin(x)}$.

Solution. Let $f(x) = x^{\sin(x)} = e^{\ln(x)\sin(x)}$. Then

$$f'(x) = e^{\ln(x)\sin(x)}(\frac{\sin(x)}{x} + \ln(x)\cos(x)) = x^{\sin(x)}(\frac{\sin(x)}{x} + \ln(x)\cos(x))$$

□

6. Find the derivative of $\sinh(x)\tanh(x)$.

Solution. Let $f(x) = \sinh(x)\tanh(x)$. Then

$$\begin{aligned} f'(x) &= \cosh(x)\tanh(x) + \sinh(x)\operatorname{sech}^2(x) \\ &= \sinh(x) + \tanh(x)\operatorname{sech}(x) \end{aligned}$$

□

7. Use differentials or a linear aproximation to estimate $\ln(.97)$.

Solution. Let $f(x) = \ln(x)$. Then

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ f'(1) &= 1 \end{aligned}$$

and so the tangent line to $f(x)$ at the point $(1, 0)$ is

$$\begin{aligned} y - 0 &= 1(x - 1) \\ y &= x - 1 \end{aligned}$$

meaning our approximation is this line evaluated at $x = .97$ so

$$\ln(.97) \approx .97 - 1 = -.03$$

□

8. Find the absolute maximum and absolute minimum values $f(x) = x^3 - 3x - 1$ on the interval $[-3, 3]$.

Solution.

$$f'(x) = 3x^2 - 3$$

so

$$\begin{aligned} f'(x) = 0 &\iff 3x^2 - 3 = 0 \\ &\iff 3x^2 = 3 \\ &\iff x^2 = 1 \\ &\iff x = \pm 1 \end{aligned}$$

Now

$$\begin{aligned} f(1) &= 1 - 3 - 1 = -3 \\ f(-1) &= -1 + 3 - 1 = 1 \\ f(3) &= 27 - 9 - 1 = 17 \\ f(-3) &= -27 + 9 - 1 = -19 \end{aligned}$$

so -19 is the absolute minimum and 17 is the absolute maximum on the interval $[-3, 3]$. □

9. Find all critical numbers of the function $f(x) = 5x^{\frac{2}{3}} + x^{\frac{5}{3}}$.

Solution.

$$f'(x) = \frac{10}{3x^{\frac{1}{3}}} + \frac{5x^{\frac{2}{3}}}{3}$$

so immediately we see that $x = 0$ is a critical point. Now we want to solve $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\iff \frac{10}{3x^{\frac{1}{3}}} + \frac{5x^{\frac{2}{3}}}{3} = 0 \\ &\iff \frac{10}{3x^{\frac{1}{3}}} = -\frac{5x^{\frac{2}{3}}}{3} \\ &\iff -2 = x \end{aligned}$$

Thus the critical points of $f(x)$ are

$$x = 0, -2$$

□

10. Show that the equation $2x - 1 - \sin(x) = 0.$ has exactly one real root.

Solution. Let $f(x) = 2x - 1 - \sin(x)$. First we show that $f(x)$ has at least one real root. Observe that $f(x)$ is a continuous function. Now

$$\begin{aligned} f(0) &= -1 < 0 \\ f(\pi) &= 2\pi - 1 > 0 \end{aligned}$$

so by the intermediate value theorem

$$\exists c \in (0, \pi) : f(c) = 0.$$

Now

$$f'(x) = 2 - \cos(x) \geq 1$$

so $f'(x)$ has no real roots. Suppose, for a contradiction, $f(x)$ has two or more real roots. By Rolle's theorem, $f'(x)$ would have a real root which contradicts the fact that $f'(x) \geq 1$ has no real roots. Thus it must be the case that $f(x)$ has at most one real root. Since we know that $f(x)$ has at least one real root, we now have that $f(x)$ has exactly one real root. □

11. Find the intervals on which f is increasing or decreasing and all local maximum and minimum values of $f(x) = 3x^{\frac{2}{3}} - x$.

Solution.

$$f'(x) = \frac{2}{x^{\frac{1}{3}}} - 1$$

so

$$\begin{aligned} f'(x) = 0 &\iff \frac{2}{x^{\frac{1}{3}}} - 1 = 0 \\ &\iff \frac{2}{x^{\frac{1}{3}}} = 1 \\ &\iff 2 = x^{\frac{1}{3}} \\ &\iff 8 = x \end{aligned}$$

and so the critical points of $f(x)$ are $x = 0, 8$. Now we solve for when $f'(x) < 0$:

$$\begin{aligned} f'(x) < 0 &\iff \frac{2}{x^{\frac{1}{3}}} - 1 < 0 \\ &\iff \frac{2}{x^{\frac{1}{3}}} < 1 \\ &\iff 2 < x^{\frac{1}{3}} \text{ or } x < 0 \\ &\iff 8 < x \text{ or } x < 0 \end{aligned}$$

and so

$$f'(x) < 0 \iff x \in (-\infty, 0) \cup (8, \infty).$$

Then the only remaining possibility is that

$$f'(x) > 0 \iff x \in (0, 8)$$

Thus f is increasing on $(0, 8)$, decreasing on $(-\infty, 0) \cup (8, \infty)$, $f(8)$ is a local maximum, and $f(0)$ is a local minimum. □

12. Find the limit $\lim_{x\to 0^+} \frac{\ln(x)}{x}$.

Solution. We do not need to apply L'Hôpital's rule. The numerator tends to $-\infty$. The bottom tends to 0 but stays positive. Thus

$$\lim_{x\to 0^+} \frac{\ln(x)}{x} = -\infty.$$

□

13. Find the limit $\lim_{x\to 0} \frac{\cos(x)-1+\frac{x^2}{2}}{x^4}$

Solution. The numerator and denominator both tend to 0 so we may apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\cos(x)-1+\frac{x^2}{2}}{x^4} = \lim_{x\to 0} \frac{-\sin(x)+x}{4x^3}$$

Again, the numerator and denominator both tend to 0 so we may apply L'Hôpital's rule another time:

$$\begin{aligned}
\lim_{x\to 0} \frac{-\sin(x)+x}{4x^3} &= \lim_{x\to 0} \frac{-\cos(x)+1}{12x^2} \\
&= \lim_{x\to 0} \frac{\sin(x)}{24x} \text{ (again we have used L'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{\cos(x)}{24} \text{ (again we have used L'Hôpital's rule)} \\
&= \frac{1}{24}.
\end{aligned}$$

□

14. Sketch the curve $y = \sqrt[3]{x^2-1}$.

Solution. The domain of $y = \sqrt[3]{x^2-1}$ is $\mathbb{R}$, the entire real line. Now

$$y' = \frac{2x}{3(x^2-1)^{\frac{2}{3}}}$$

so immediately we see that the critical points are $x = 0, \pm 1$. Notice that the denominator is always positive since

$$(x^2-1)^{\frac{2}{3}} = (\sqrt[3]{x^2-1})^2$$

meaning the sign of y' is determined by the numerator, $2x$. Thus

$$\begin{aligned}
y' > 0 &\iff 0 < x < 1 \text{ and } 1 < x \\
y' < 0 &\iff x < -1 \text{ and } -1 < x < 0.
\end{aligned}$$

Thus $\sqrt[3]{0^2-1}=-1$ is a local minimum. We also see that y is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. y clearly has zeroes at $x=\pm 1$. For large $|x|$, we have that

$$x^2-1\sim x^2 \text{ so } \sqrt[3]{x^2-1}\sim\sqrt[3]{x^2}=x^{\frac{2}{3}}.$$

The function obtains a local minimum when $x=0$ as we observed earlier. The function is not differentiable at $x=\pm 1$. In fact, from our formula for the derivative, it is easy to see that

$$\lim_{x\to -1} y' = \lim_{x\to -1}\frac{2x}{3(x^2-1)^{\frac{2}{3}}} = -\infty$$
$$\lim_{x\to 1} y' = \lim_{x\to 1}\frac{2x}{3(x^2-1)^{\frac{2}{3}}} = \infty.$$

Click here to see the graph. Zoom in/out as necessary. □

15. Sketch the curve $y=\frac{\ln(x)}{x}$ for $x>0$.

Solution. The domain of y is $\mathbb{R}_{>0}$, the positive real axis. Now

$$y'=\frac{x\cdot\frac{1}{x}-\ln(x)}{x^2}=\frac{1-\ln(x)}{x^2}.$$

We proceed to solve $y'=0$:

$$\begin{aligned} y'=0 &\iff \frac{1-\ln(x)}{x^2}=0\\ &\iff 1-\ln(x)=0\\ &\iff 1=\ln(x)\\ &\iff e=x \end{aligned}$$

so the only critical point is $x=e$. Again, notice that the denominator of the derivative is always postive so the sign of the derivative is determined by the sign of the numerator. Thus

$$y'>0\iff 1-\ln(x)>0\iff 1>\ln(x)\iff x<e$$
$$y'<0\iff 1-\ln(x)<0\iff 1<\ln(x)\iff e<x$$

whence y is decreasing on (e,∞), increasing on $(0,e)$. Moreover, $\frac{\ln(e)}{e}=\frac{1}{e}$ is a local maximum. The function has a zero at $x=1$. Now we want to compute $\lim_{x\to\infty}\frac{\ln(x)}{x}$:

$$\begin{aligned} \lim_{x\to\infty}\frac{\ln(x)}{x} &= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} \qquad \text{(by L'Hôpital's rule)}\\ &= \lim_{x\to\infty}\frac{1}{x}\\ &= 0. \end{aligned}$$

Next, we compute $\lim_{x\to 0^+}\frac{\ln(x)}{x}=-\infty$ (this was problem #12). The function is differentiable for every $x\in\mathbb{R}_{>0}$. Click here to see the graph. Zoom in/out as necessary. □