## NOTES ON STOKES' THEOREM

In class, I gave / tried to give two examples, but because of time constraints and in-presentation errors, we finished neither. I wanted to make sure you had the chance to see both arguments in full. I did draw all the necessary pictures in class, and so you can refer to your notes for those; I won't reproduce them here.

### 1. First example

Consider the vector field $\vec{F} = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$, and unit circle $C$ centered at the origin. By picking the parametrization $C(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, we can compute $\oint_C F \cdot dr$ as follows:

$$\begin{aligned} \oint_C F \cdot dr &= \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} \frac{-\sin t}{\cos^2 t + \sin^2 t} \\ \frac{\cos t}{\cos^2 t + \sin^2 t} \end{pmatrix} dt \\ &= \int_0^{2\pi} 1 \cdot dt = 2\pi. \end{aligned}$$

One thing that we spent time talking about in class is that Green's theorem does not apply to this integral. The problem is that the vector field is not defined at the origin, and so we can't perform the double-integral over the interior of the disk. Let's instead consider the 3-dimensional vector field

$$\vec{G} = \begin{pmatrix} \frac{-y}{x^2+y^2+z^2} \\ \frac{x}{x^2+y^2+z^2} \\ 0 \end{pmatrix}.$$

If we set $z = 0$, so we're working in the $xy$-plane, we recover the vector field $F$. We stuck the $z^2$ into the denominator to ensure that $G$ is also only undefined at the origin, but it has a good value everywhere else. In 3-dimensional space, unlike in 2-dimensional space, we can draw a surface whose boundary is the circle and which misses the origin: for example, we can let the surface $\Sigma$ be the hemisphere of radius 1, centered at the origin, with $b(\Sigma) = C$. Then, Stokes' theorem asserts

$$\oint_C F \cdot dr = \oint_C G \cdot dr = \iint_\Sigma \nabla \times G \cdot dS.$$

Let's test Stokes' theorem in this case by computing the right-most expression.

To do that, we have to parametrize the surface: adapting spherical coordinates to our situation gives

$$\Sigma(\phi, \theta) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \frac{\pi}{2}.$$

Then, $dS$ is given by the formula

$$dS = \frac{\partial \Sigma}{\partial \phi} \times \frac{\partial \Sigma}{\partial \theta} = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix} \times \begin{pmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \sin\phi\cos\phi \end{pmatrix}.$$

We can quickly check that this is the outward-facing orientation. We also have to compute $\nabla \times G$:

$$\nabla \times G = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} \frac{-y}{x^2+y^2+z^2} \\ \frac{x}{x^2+y^2+z^2} \\ 0 \end{pmatrix} = \frac{1}{(x^2+y^2+z^2)^2} \cdot \begin{pmatrix} 2xz \\ 2yz \\ 2z^2 \end{pmatrix} = \begin{pmatrix} 2\sin\phi\cos\phi\cos\theta \\ 2\sin\phi\cos\phi\sin\theta \\ 2\cos^2\phi \end{pmatrix}.$$

Then,

$$\iint_\Sigma \nabla\times G\cdot dS = \int_0^{2\pi}\int_0^{\pi/2} \begin{pmatrix} 2\sin\phi\cos\phi\cos\theta \\ 2\sin\phi\cos\phi\sin\theta \\ 2\cos^2\phi \end{pmatrix}\cdot\begin{pmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \sin\phi\cos\phi \end{pmatrix} d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/2} 2\cdot(\sin^3\phi\cos\phi\cos^2\theta + \sin^3\phi\cos\phi\sin^2\theta + \sin\phi\cos^3\phi)d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/2} 2\cdot\sin\phi\cos\phi d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/2} \sin(2\phi)d\phi d\theta = 2\pi.$$

So, we see that they match. The moral of this example is that Stokes' theorem applies where Green's theorem gets stuck. Namely, moving a point from the center of a planar figure causes it to become non-simply-connected, whereas a simply connected solid with a point removed from the center remains simply connected.

## 2. Example 2

Now, let's compute the line integral $\oint_C F\cdot dr$, where $F$ is defined by

$$\vec{F} = \begin{pmatrix} y^3 \\ x \\ z \end{pmatrix}$$

and $C$ is implicitly given by the pair of constraints

$$C = \left\{ \begin{array}{c} x^2+y^2=1, \\ z=2+\sin(xy) \end{array} \right\},$$

oriented clockwise. The first constraint is the equation of a cylinder, and the second is the equation of some wavy plane, so their intersection is a wavy circle, hovering above the unit circle in the $xy$-plane. It's possible to parametrize this by taking $x=\cos t$ and $y=\sin t$, then but that yields $z=2+\sin(\sin t\cdot\cos t)$, which looks pretty impossible to integrate.

Instead, we will apply Stokes' theorem to $\Sigma$, the part of the cylinder caught between $C$ and the unit circle $D$ in the $xy$-plane. If we orient this cylinder to have an outward-facing normal and $D$ to also be clockwise oriented, then Stokes' theorem states

$$\oint_C F\cdot dr = \iint_\Sigma \nabla\times F\cdot dS + \oint_D F\cdot dr.$$

So, to compute the left-hand side, we are tasked with computing the two integrals on the right. The surface integral is easy: we compute

$$\nabla\times F = \begin{pmatrix} 0 \\ 0 \\ 1-3y^2 \end{pmatrix}.$$

On the other hand, $dS$ on $\Sigma$ is always a positive multiple of the outward-facing normal, which has no $z$-component, and hence the dot product $(\nabla\times F)\cdot dS$ vanishes identically and $\iint_\Sigma \nabla\times F\cdot dS = 0$.

What's left, then, is to compute the line integral around $D$. Unlike $C$, $D$ is very simple to parametrize:

$$D(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix},\ D'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}.$$

So, we compute:

$$\begin{aligned}
\oint_D F\cdot dr &= \int_0^{2\pi} \begin{pmatrix} \sin^3 t \\ \cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt \\
&= \int_0^{2\pi} (\cos^2 t - \sin^4 t) dt = \int_0^{2\pi} (\cos^2 t - \sin^2 t \cdot (1-\cos^2 t)) dt \\
&= \int_0^{2\pi} \left(\cos(2t) + \frac{\sin^2(2t)}{4}\right) dt = \int_0^{2\pi} \left(\cos(2t) + \frac{\cos(2t)-1}{8}\right) dt \\
&= \left[\frac{\sin(2t)}{2} + \frac{\sin(2t)}{16} - \frac{t}{8}\right]_0^{2\pi} = \frac{\pi}{4}.
\end{aligned}$$

Hence,

$$\oint_C F\cdot dr = \iint_\Sigma \nabla\times F\cdot dS + \oint_D F\cdot dr = 0 + \frac{\pi}{4} = \frac{\pi}{4}.$$

The moral here is in the application of Stokes' theorem: it tells you that if you drag your curve through space, tracing out a surface behind it, then the line integrals around the start and end loops are related to each other by a correction factor of this flux integral.