MATH1B, FALL 2010. MIDTERM 1 SOLUTION

Multiple choice: 1.D, 2.A, 3.C, 4.E, 5.C, 6.A, 7.D, 8.D.

Problem 1

(a) Since the degree of the numerator equals that of the denominator, we must use long division, or just rewrite the integrand

$$\frac{x^2}{4x^2+4x+10} = \frac{1}{4}\frac{4x^2}{4x^2+4x+10} = \frac{1}{4}\frac{4x^2+4x+10-4x-10}{4x^2+4x+10} = \frac{1}{4} - \frac{1}{4}\frac{4x+10}{4x^2+4x+10}$$

The discriminant of the denominator is negative, so partial fractions is unnecessary. Instead we just manipulate a bit. Ignoring the first term for the moment, and noticing that the derivative of the denominator is $8x+4$, we rewrite the numerator of the second term:

$$\frac{1}{4}\frac{4x+10}{4x^2+4x+10} = \frac{1}{8}\frac{8x+20}{4x^2+4x+10} = \frac{1}{8}\frac{8x+4}{4x^2+4x+10} + \frac{1}{8}\frac{16}{4x^2+4x+10}$$

Finally, we can complete the square of the denominator: $4x^2+4x+10 = (2x+1)^2+3^2$. Putting all this back together, we have:

$$\begin{aligned}\int \frac{x^2}{4x^2+4x+10}dx &= \int \frac{1}{4}dx - \frac{1}{8}\int \frac{8x+4}{4x^2+4x+10}dx - \int \frac{2}{(2x+1)^2+3^2}dx \\ &= \frac{1}{4}x - \frac{1}{8}\ln(4x^2+4x+10) - \frac{1}{3}\arctan\frac{2x+1}{3} + C.\end{aligned}$$

(b) Make the substitution $x = \sqrt{t}$, so $t = x^2$, and $dt = 2xdx$. Then our integral looks like

$$\begin{aligned}\int x^2 e^x (2x)dx &= 2\int x^3 e^x dx \\ &= 2\left[x^3e^x - 3\int x^2 e^x dx\right] \\ &= 2\left[x^3e^x - 3\left[x^2e^x - 2\int xe^x dx\right]\right] \\ &= 2\left[x^3e^x - 3\left[x^2e^x - 2\left[xe^x - \int e^x dx\right]\right]\right] \\ &= 2e^x\left[x^3 - 3x^2 + 6x - 6\right] + C \\ &= 2e^{\sqrt{t}}\left[t^{3/2} - 3t + 6\sqrt{t} - 6\right] + C.\end{aligned}$$

(c) Make the substitution $u = \ln(\tan x)$, so that $du = \frac{1}{\tan x}\sec^2 x\,dx = \frac{1}{\sin x\cos x}dx$. When $x = \pi/4$, $u = \ln 1 = 0$, and when $x = \pi/3$, $u = \ln\sqrt{3}$. Now our integral looks like

$$\int_0^{\ln\sqrt{3}} u\,du = \frac{1}{2}\left[u^2\right]_0^{\ln\sqrt{3}} = \frac{1}{2}(\ln\sqrt{3})^2 = \frac{1}{2}\left(\frac{1}{2}\ln 3\right)^2 = \frac{1}{8}(\ln 3)^2.$$

Problem 2

(a) This integral is improper since it is taken over an unbounded domain. To check that there are no other problems, make sure the denominator is not undefined or zero: $\cos x + \sin x > 0$ for $0 \le x < 3\pi/4$, so $\cos x + \sin x + x^6 > 0$ for $0 \le x < 3\pi/4$. On the other hand, when $x \ge 3\pi/4$, we have $x^6 > 2$ and $\cos x + \sin x > -2$ (this is true for every x), so $\cos x + \sin x + x^6 > 0$ when $x > 3\pi/4$.

Thus the integrand is defined for all $0 \le x \le \infty$, so the convergence of the integral depends only on what happens near ∞. Roughly, the idea is that for large x, we can ignore the $\cos x + \sin x$ in the denominator, so the integrand looks like $\frac{x}{\sqrt{x^6}} = \frac{1}{x^2}$, which converges. To make this precise, we want to use the comparison test. The only problem is that the integral of $1/x^2$ doesn't actually converge near zero. This isn't really a problem, we just have to break up the integral into two pieces:

$$\int_0^\infty \frac{x}{\sqrt{\cos x + \sin x + x^6}} dx = \int_0^2 \frac{x}{\sqrt{\cos x + \sin x + x^6}} dx + \int_2^\infty \frac{x}{\sqrt{\cos x + \sin x + x^6}} dx$$

The integrand is continuous for $x \ge 0$, so the first piece is finite, and we can ignore it. For the second integral, the tricky part is that since $\cos x + \sin x$ is sometimes positive, sometimes negative, we *cannot* say that $\cos x + \sin x + x^6 > x^6$. We must be just a little bit more careful: for any x, we have $-2 < \sin x + \cos x$, which means $x^6 - 2 < \cos x + \sin x + x^6$. Also, when $x \ge 2$, then certainly $x^6 > 4$, so $\frac{1}{4}x^6 < x^6 - 2$, which implies that $\frac{1}{4}x^6 < \cos x + \sin x + x^6$. So

$$\frac{x}{\sqrt{\cos x + \sin x + x^6}} < \frac{x}{\sqrt{\frac{1}{4}x^6}} = \frac{2}{x^2}$$

Since $\int_2^\infty \frac{2}{x^2}\, dx$ converges, so does $\int_2^\infty \frac{x}{\sqrt{\cos x + \sin x + x^6}}\, dx$, by the comparison test.

(b) This integral is improper because the integrand is undefined at $x = 0$. So we must rewrite it as

$$\int_0^{\pi/2} \frac{1}{1 - e^x} dx = \lim_{t \to 0^+} \int_t^{\pi/2} \frac{1}{1 - e^x} dx$$

To compute this integral, we set $u = e^x$, so $x = \ln u$ and $dx = 1/u\ du$. Ignoring the limits of integration briefly, our integral becomes

$$\int \frac{1}{1-u}\frac{1}{u} du = \int \frac{1 - u + u}{(1-u)u} du = \int \left[\frac{1}{u} + \frac{1}{1-u}\right] du = \ln u - \ln|1-u| = \ln|\frac{u}{1-u}|$$

Now we use this to compute the improper integral:

$$\lim_{t \to 0^+} \int_t^{\pi/2} \frac{1}{1 - e^x} dx = \lim_{t \to 0^+} \left[\ln|\frac{e^x}{1 - e^x}|\right]_t^{\pi/2}$$

Since $\lim_{t \to 0^+} \frac{e^t}{1-e^t} = \infty$, and ln is increasing, $\lim_{t \to 0^+} \ln\left[\frac{e^t}{1-e^t}\right] = \infty$, so the integral diverges.

Problem 3

(a) Let $S_N = \Sigma_{n=1}^N a_n = a_1 + a_2 + \ldots + a_N$. Then $\Sigma_{n=1}^\infty a_n$ converges if $\lim_{N\to\infty} S_N = L < \infty$.

(b) We must find an explicit formula for the sequence of partial sums. Using partial fractions,

$$\frac{1}{n^2 - n - 2} = \frac{1}{(n-2)(n+1)} = \frac{1/3}{n-2} - \frac{1/3}{n+1}, \text{ so}$$

$$S_N = \frac{1}{3}\left(\left[\frac{1}{2} - \frac{1}{5}\right] + \left[\frac{1}{3} - \frac{1}{6}\right] + \left[\frac{1}{4} - \frac{1}{6}\right] + \left[\frac{1}{5} - \frac{1}{7}\right] + \left[\frac{1}{6} - \frac{1}{8}\right] \ldots + \left[\frac{1}{N-2} - \frac{1}{N+1}\right]\right)$$

All the positive terms except the first three cancel out all but the last three negative terms, leaving

$$S_N = \frac{1}{3}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{N-1} - \frac{1}{N} - \frac{1}{N+1}\right]$$

(serious skeptics should prove this by induction). Now take the limit:

$$\lim_{N\to\infty} S_N = \frac{1}{3}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right] = \frac{13}{36}$$

According to the definition in (a), this means in particular that Σa_n converges.