STEWART'S CALCULUS, EXERCISE 8.2.27

Fixing some positive parameter a, we're given the implicit curve

$$3ay^2 = x(a-x)^2,$$

and we're asked to find the surface area of the surface of rotation part of it sweeps out by rotating first along the x-axis [part a)] and then along the y-axis [part b)].

a) The given curve is vertically symmetric; if (x, y) is a solution, then so is $(x, -y)$, since $3a(-y)^2 = 3ay^2$. Hence, the self-intersection points of the curve (which bound the loop we're interested in) occur when $y = 0$, and solving $0 = x(a-x)^2$ shows the only solutions to be $x = 0$ and $x = a$. Additionally, this means that the part of the curve contained in the upper half-plane will sweep out the whole surface when rotated about the x-axis, so when we solve[1] for y it's OK to restrict to positive solutions:

$$y = \sqrt{\frac{x(a-x)^2}{3a}} = (a-x)\sqrt{\frac{x}{3a}}.$$

The derivative of y is also easy to compute:

$$\begin{aligned}\frac{dy}{dx} &= -1 \cdot \sqrt{\frac{x}{3a}} + (a-x)\frac{1}{2\sqrt{3ax}} \\ &= \frac{a-3x}{2\sqrt{3ax}}.\end{aligned}$$

With all this in hand, we can start computing the arclength integral:

$$\begin{aligned}S &= \int_0^a 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx \\ &= \int_0^a 2\pi(a-x)\sqrt{\frac{x}{3a}} \cdot \sqrt{1 + \left(\frac{a-3x}{2\sqrt{3ax}}\right)^2}dx \\ &= \int_0^a 2\pi(a-x)\sqrt{\frac{x}{3a}} \cdot \sqrt{\frac{1}{12ax}} \cdot \sqrt{12ax + (a-3x)^2}dx \\ &= \int_0^a 2\pi(a-x)\frac{1}{6a} \cdot \sqrt{a^2 + 6ax + 9x^2}dx \\ &= \frac{\pi}{3a}\int_0^a (a-x)(a+3x)dx \\ &= \frac{\pi}{3a}\left[a^2x + ax^2 - x^3\right]_0^a \\ &= \frac{\pi a^2}{3}.\end{aligned}$$

b) The idea with computing the area of the surface of rotation about the y-axis is to reuse as much of our data as possible. The general formula for surface area under rotation about the y-axis looks like

$$S = \int_{\cdots}^{\cdots} 2\pi x d\ell,$$

[1] In class, my not-so-bright idea was to use implicit differentiation, which gives $y' = \frac{(a-x)^2 - 2x(a-x)}{6ay}$. This has the attractive feature of cancelling with the y that appears outside the radical in the arclength integral, but it turns out that the numerator is too abysmal to push the rest of the integral through. There are lots of similarly innocuous but incorrect ways to approach this integral, but this is the only one I see that turns out to be tractable.

where $d\ell$ is the arclength differential discussed in class. We can, in particular, select

$$d\ell = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

just as before, and use this $d\ell$ to build the surface area integral[2].

This allows us to reuse the derivative we've already computed. It also tells us where our bounds are, since they're determined by the same two self-intersection points as before. — But this time, that we've selected the positive branch of y matters, since we're rotating about the y-axis. There's another branch of our curve sitting below the x-axis which is so far unaccounted for, but because it's a mirror image of the positive branch we can double our end result to produce the total surface area.[3]

So, we have

$$S = 2\int_0^a 2\pi x\sqrt{1 + \left(\frac{a-3x}{2\sqrt{3ax}}\right)^2}\, dx,$$

which we start to compute just as before:

$$\begin{aligned} S &= 2\int_0^a 2\pi x\sqrt{\frac{a^2+6ax+9x^2}{12ax}} dx \\ &= 2\int_0^a 2\pi x \cdot \frac{a+3x}{2\sqrt{3ax}} dx \\ &= \frac{2\pi}{\sqrt{3a}} \int_0^a \left(a\sqrt{x} + 3x\sqrt{x}\right) dx \\ &= \frac{2\pi}{\sqrt{3a}} \left[\frac{2a}{3}x^{3/2} + \frac{6}{5}x^{5/2}\right]_0^a \\ &= \frac{2\pi}{\sqrt{3a}} \left(\frac{2a^{5/2}}{3} + \frac{6a^{5/2}}{5}\right) \\ &= \frac{56\pi a^2}{15\sqrt{3}}. \end{aligned}$$

[2] This works well despite the surprising — for me, at least! — feature that this will result in an integral along the x-axis.

[3] In general, you'll want to solve for equations for both branches, compute a surface area integral for each, and sum them.