TAYLOR TOWERS FOR HOMOTOPY FUNCTORS

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ABSTRACT. Just as one can define derivatives and approximating polynomials for smooth functions on spaces with smooth structure, there is a wholly analogous construction for certain functors between model categories with certain extra properties. We define these objects, investigate some simple examples, and consider an associated spectral sequence. These are talk notes given in the xkcd seminar at Stanford in February 2011, then again at Uni–Bonn in April 2012.

1. SECANT AND TANGENT CURVES

Before we get started on talking about finding polynomial approximations to functors, let's spend a few minutes revisiting the story for smooth functions on the real line. Differential calculus begins with the following construction: select a function f, a special point $x_0 \in \mathbb{R}$, and some other point $x_1 \in \mathbb{R}$. The *secant line* corresponding to this data is the unique line interpolating the pairs $(x_0, f(x_0))$ and $(x_1, f(x_1))$, for which we can write down the equation

$$T_1 f = f(x_1) \cdot \frac{x - x_0}{x_1 - x_0} + f(x_0) \cdot \frac{x - x_1}{x_0 - x_1}.$$

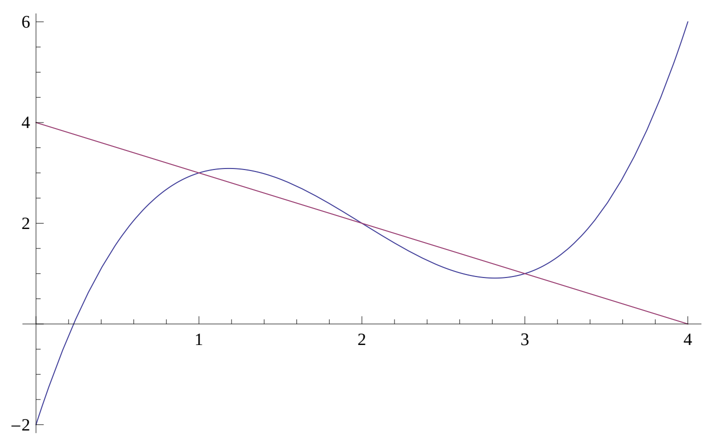

FIGURE 1. The secant line through 2 and 3 on a cubic.

Then, when we bring *limits* into the picture. Letting x_0 and x_1 tend toward 0, we find $(P_1 f)(x)$, the linear (i.e., first order) approximation to f at 0.

Of course, it is possible to build interpolating polynomials through as many points as we'd like: for any set of $(n+1)$ points in the plane that share no x-coordinates among them, there is a unique interpolating polynomial of degree n that passes through each of them. The formula given above for the interpolating line generalizes to Lagrange's formula:

$$y = \sum_{i=0}^{n} f(x_i) \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

As an example, let's set $n = 2$, so that we build "secant parabolas," and then pick $f(x) = e^x$ along with the points with x-values h, 0, and $-h$ to test. Just as before, we can let these three points cluster toward 0 to attempt to build

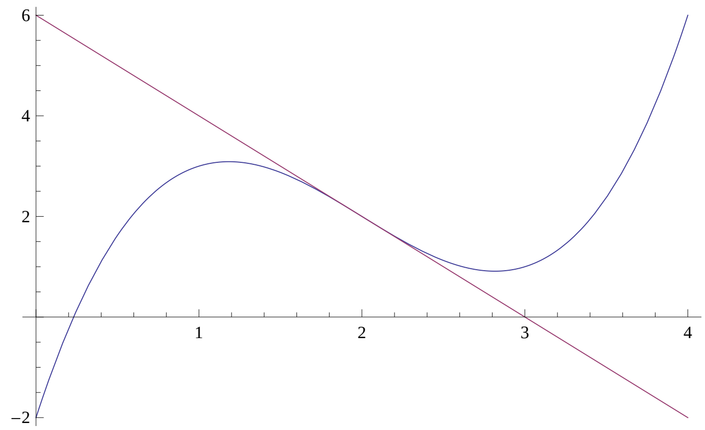

FIGURE 2. The tangent line through 2 on a cubic.

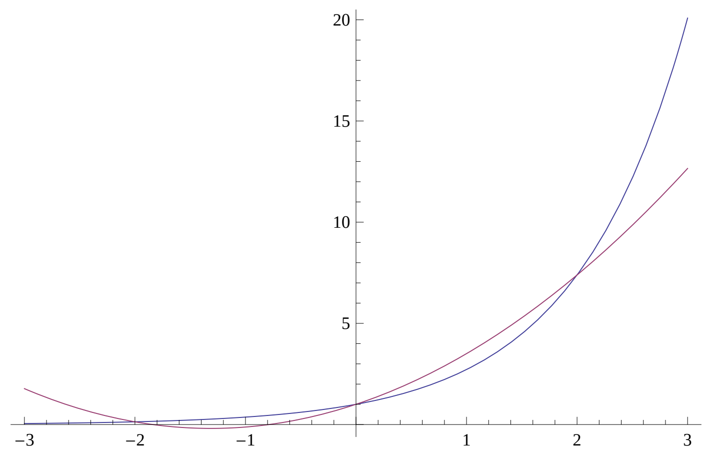

FIGURE 3. The secant parabola through $h = 2$, 0, and -2 on the exponential.

a tangent parabola — in the example, this means taking the limit $h \to 0$. If we expand out the Lagrange formula above, we get

$$\begin{aligned}
(P_2 \exp)(x) &= \lim_{h\to 0} (T_2 \exp)(x)) \\
&= \lim_{h\to 0} \left(\frac{e^h - 2 + e^{-h}}{2h^2} \cdot x^2 + \frac{e^h - e^{-h}}{2h} + 1 \right) \\
&= \left(\lim_{h\to 0} \frac{e^h - 2 + e^{-h}}{2h^2} \right) \cdot x^2 + \left(\lim_{h\to 0} \frac{e^h - e^{-h}}{2h} \right) \cdot x + \left(\lim_{h\to 0} 1 \right) \cdot 1.
\end{aligned}$$

Each of these limits can individually be calculated to be $\frac{1}{2}$, 1, and 1, giving $(P_2^0 \exp)(x) = \frac{1}{2}x^2 + x + 1$.

Now it's time to get excited, since you recognize this polynomial from elsewhere. Calculus students studying Taylor series learn the formula

$$(P_n f)(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)x^i}{i!},$$

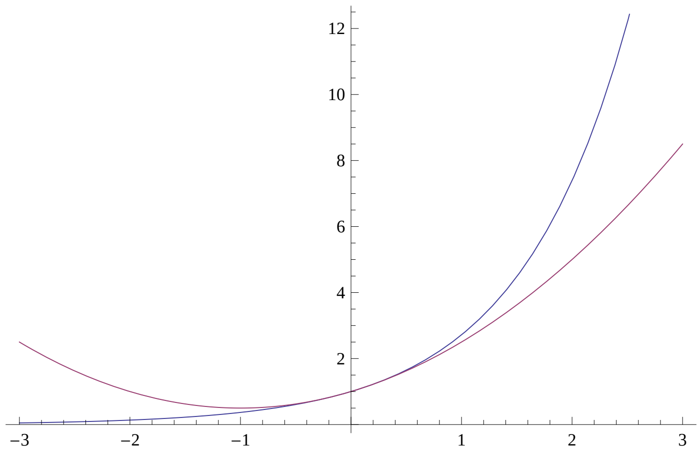

FIGURE 4. The tangent parabola at 0 on the exponential.

and using this one finds that the beginning of the expansion for the exponential function looks like $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$, exactly matching what we found above. It turns out that this is not an accident — for a smooth function f, these two definitions of $P_n f$ coincide.[1]

These two approaches have their merits and dismerits. What's nice about the summation definition is that it turns out to be very computable; we have an extremely successful theory for computing the global derivatives of common smooth functions. What's nice about the geometric definintion is that it requires very little added machinery — specifically, we made decisions about what "polynomial interpolation" and "limit" should mean, then approximations of all orders immediately followed. This means that it is *portable* in a sense very important to us. We previously heard about linear functors, which means in this talk we should be all set to talk about polynomial approximations of higher order.[2]

Summation formula	Geometric definition
Some clear properties: $P_n P_{n+k} = P_n$	Moral value
Requires (iterated) derivatives	Awkward to compute
Computable	*Portable*

2. INTERPOLATION FOR FUNCTORS

Now we're going to rephrase this set-up to give a differential calculus of functors, so our goal are:

(1) Decide what the words "interpolating polynomial" and "limit" mean.
(2) Figure out how to evaluate $P_n F$ away from the basepoint, at an arbitrary X.

From here on, we'll fix a source ∞-category C and a target ∞-category D, and consider functors $F : \mathsf{C} \to \mathsf{D}$. We'll want…

- …finite colimits to exist in C.
- …for C to have a final object. This will be our notion of "basepoint."
- …finite limits and directed colimits to exist in D.
- …for these finite limits and directed colimits in D to commute.

[1] I learned this analogy-crucial fact from Randy McCarthy. As an unimportant side remark, he memorably described the limit $x_0, \dots, x_n \to 0$ as "crashing toward the basepoint."

[2] To explain the moral value in the table, Taylor polynomials are often justified to students by drawing their graphs and noticing that they look quite similar to the graphs of the original functions near their centers. This is actually built in to the geometric construction, rather than a "Hmm, that's curious." side-remark.

That I'm picking an ∞-categorical setting is mostly a matter of abbreviation; the only real homotopy theory we'll use are facts about homotopy co/limits, which are entirely equivalent to ∞-categorical co/limits, and this way I am free to forget to say "homotopy" before "colimit" without causing catastrophe.

Recall that a linear functor (or a functor that is polynomial of degree 1, or a 1-excisive functor) is one that carries homotopy pushout squares to homotopy pullback squares. There is an obvious mode of generalization here: a homotopy pushout square models the decomposition of the pushout into 2 spaces, with their possibly nontrivial intersection marked at the opposite corner. Let's replace the pushout square in this set-up with a sort of pushout hypercube, trading a 2-parameter condition defining a degree 1-polynomial for an $(n+1)$-parameter condition defining a degree n polynomial.

To this end, let's briefly recount the definitions of Cartesian, co-Cartesian, and strongly co-Cartesian:

- A cubical diagram $\mathscr{X}$ of dimension n is a diagram indexed by the lattice of subsets $T \subseteq S$ of a finite set S of cardinality n, i.e., it is indexed by the partially ordered powerset $\mathscr{P}S$.
- Let $\mathscr{P}_0 S$ denote the full subcategory of $\mathscr{P}S$ of subsets of positive cardinality. An n-cube $\mathscr{X}$ is said to be Cartesian if the limit of $\mathscr{X}$ restricted to $\mathscr{P}_0 S$ agrees with $\mathscr{X}(\emptyset)$.
- An n-cube $\mathscr{X}$ is said to be co-Cartesian if it satisfies the dual condition. Let $\mathscr{P}_1 S$ denote the full subcategory of $\mathscr{P}S$ of proper subsets of S. Then, $\mathscr{X}$ is co-Cartesian if the colimit of the restriction of $\mathscr{X}$ to $\mathscr{P}_1 S$ agrees with $\mathscr{X}(S)$.
- Finally, an n-cube $\mathscr{X}$ indexed by subsets of a set S is said to be strongly co-Cartesian if for every choice of $T \subseteq S$ with $|T| > 1$, the restriction of $\mathscr{X}$ to subsets of T gives a co-Cartesian cube. Equivalently, $\mathscr{X}$ is strongly co-Cartesian when it is the Kan extension of its restriction to the verticesof cardinality 1.

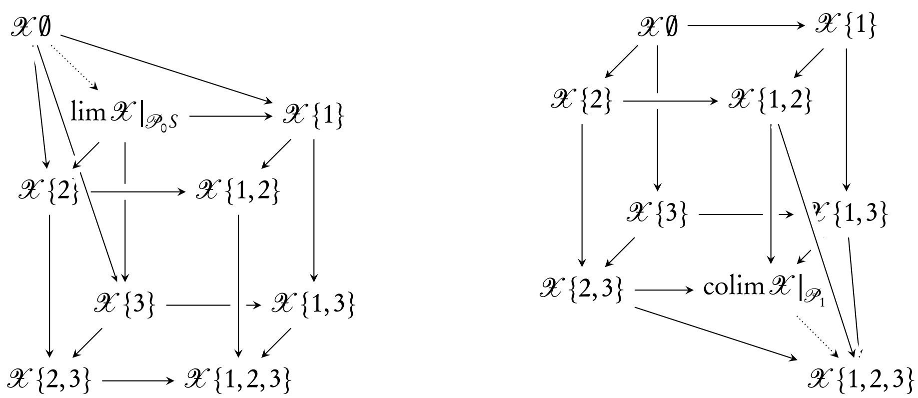

FIGURE 5. The Cartesian and co-Cartesian conditions for cubes indexed by $S = \{1,2,3\}$.

We now have the language for our major definition: F is polynomial of degree n (or n-excisive) if it takes strongly co-Cartesian $(n+1)$-cubes to Cartesian $(n+1)$-cubes. This is meant to be in direct analogy with the classical situation, where Lagrange's formula tells you that if you know the value of a polynomial of degree n at $(n+1)$ sample points, then you can reconstruct the whole thing and sample it at any other point you choose. If we build a strongly co-Cartesian $(n+1)$-cube around a certain space X, then F is degree n if the value of F on the rest of the cube is enough to recover FX through this fixed method of interpolation.

Two remarks are in order. First, choosing strongly co-Cartesian over merely co-Cartesian is important, because we want an analogue of the statement that polynomials of order n are also of order m for $m \geq n$. The proof of this came up in the previous talk. Second, let's check an edge case. Intuitively, if F is polynomial of degree 0, then it ought to be (locally) constant, based on our experience with real functions. By our definitions, such a functor F takes strongly co-Cartesian 1-cubes to Cartesian 1-cubes. A 1-cube is merely an arrow, and the condition for strong co-Cartesianness is vacuous, so all arrows count as strongly co-Cartesian 1-cubes. Then, the image of any arrow under F must be Cartesian, meaning that the source of the arrow must be weakly equivalent to the limit of the diagram picking out the target of the arrow — but the limit of a one object, one arrow diagram is the object itself, and hence F must take all arrows to weak equivalences.

Getting back to it, there is one such strongly co-Cartesian cube with particularly good properties: fixing a space X, let $\mathscr{X}$ be the cube indexed by subsets $T \subseteq S$, $|S| = n+1$, whose vertex at T is the join of X and T — i.e., the $|T|$-pointed cone on X. Again, if F were polynomial of degree n, then $F(X)$ would be equivalent to the limit of the punctured cube $F \circ \mathscr{X}|_{\mathscr{P}_0 S}$, but at the very least we can record what F "ought to be" by setting

$$(T_n F)(X) = \lim F \circ \mathscr{X}|_{\mathscr{P}_0 S}.$$

This comes with a natural map $(t_n F) : F \to T_n F$ by universality of the limit.

Let's pause for a moment to further our analogy, though we'll have to restrict our setup a bit so that C has a sensible notion of homotopy groups, as for $\mathsf{C} = \mathsf{Spaces}$ or $\mathsf{C} = \mathsf{Spectra}$. Recall that $1 \in \mathsf{C}$ is the "center" of our construction, and that each object X comes with a map $X \to 1$. The connectivity k of this map measures the similarity of X to 1, and we think of the reciprocal $\frac{1}{k}$ as measuring their "nearness". If $X \to 1$ is k-connected, then notice that the objects $\mathscr{X}|_{\mathscr{P}_0 S}(R) \to 1$ are all at least $(k+1)$-connected. If our construction is supposed to be working toward "Taylor expanding around the basepoint" and we take "the basepoint" here to mean the final object, then studying $F\mathscr{X}|_{\mathscr{P}_0 S}$ means approximating the value of F at X by interpolating by values closer to the basepoint than X itself.

Of course, in the classical setup with secants, it wasn't sufficient to merely pick interpolation points nearer than the point at which you wanted to sample, there was an extra limiting step where we let the interpolation points cluster at the base. The same is true here: $T_n F$ does not have to be n-excisive, but it is "better," as its action on k-connected objects is determined by F's action on $(k+1)$-connected objects. Our analogue of clustering at the base is to iterate this construction: by applying T_n successively, we build a sequence

$$F \xrightarrow{t_n F} T_n F \xrightarrow{t_n(T_n F)} T_n T_n F \to \cdots \to P_n F,$$

yielding a functor in the colimit whose action is determined by the value of F on "very connected objects." This functor $P_n F$ also comes with a natural map $p_n F : F \to P_n F$, and it will be our analogue of the Taylor polynomial of degree n.

3. Properties of T_n and P_n

Since T_n and P_n are defined in terms of each other and of co/limits, some basic facts about co/limits produce a variety of interaction properties of these functors.

- Because T_n is exactly defined to use our interpolation scheme to guess what $F(X)$ would be if F were n-excisive, when F actually *is* n-excisive it guesses correctly. So, for n-excisive F, $t_n F : F \to T_n F$ is a weak equivalence. In turn, when F is n-excisive, $p_n F : F \to P_n F$ is also a weak equivalence.
- We've assumed that finite limits and sequential colimits in our target category commute. Our functors T_n and P_m are exactly defined in terms of finite limits and sequential colimits, so we have the commutation law $T_n P_m = P_m T_n$. In particular, this means that $T_n P_n F = P_n T_n F = P_n F$, and so at the very least $P_n F$ behaves as though it were n-excisive when we check the particular interpolation scheme used to build T_n. It also means that P_n and T_n preserve fiber sequences, which are themselves defined by a limit condition.
- In fact, $P_n F$ is *actually* n-excisive! There is a technical lemma used to show this, which for now we will state rather than prove: for any strongly co-Cartesian $(n+1)$-cube $\mathscr{X}$, the map of cubes $(t_n F\mathscr{X}) : F(\mathscr{X}) \to T_n F\mathscr{X}$ factors as $F(\mathscr{X}) \to \mathscr{Y} \to T_n F\mathscr{X}$, where $\mathscr{Y}$ is a Cartesian cube. The construction of $\mathscr{Y}$ is not obvious, and its existence is why we picked the cube of cones rather than some other cube. I'll go through the proof at the end of the talk if there's time; if not, Rezk provides a slick proof of this fact. In any event, once we have $\mathscr{Y}$, then for any strongly co-Cartesian cube $\mathscr{X}$, $P_n\mathscr{X}$ is defined as the directed colimit of

$$\begin{tikzcd}
F\mathscr{X} \arrow[rr] \arrow[dr] & & T_n F\mathscr{X} \arrow[rr] \arrow[dr] & & T_n^2 F\mathscr{X} \arrow[rr] \arrow[dr] & & \cdots \\
& \mathscr{Y}_0 \arrow[rr] \arrow[ur] & & \mathscr{Y}_1 \arrow[rr] \arrow[ur] & & \mathscr{Y}_2 \arrow[r] \arrow[ur] & \cdots .
\end{tikzcd}$$

  The colimit of the bottom row is the sequential colimit of Cartesian cubes, which is a condition about finite limits, so the result is itself Cartesian. Hence, $P_n F$ converts strongly co-Cartesian $(n+1)$-cubes to Cartesian ones, so is n-excisive. Using this, we also get a map $P_{n+k} F \to P_n F$ by applying P_{n+k} to $p_n F : F \to P_n F$, then since P_n is n-excisive and hence $(n+k)$-excisive, $P_{n+k} P_n F \simeq P_n F$.

- Let Fun denote the ∞-category of all functors $\mathsf{C} \to \mathsf{D}$, and let Exc^n denote the full subcategory of such functors which are n-excisive. The functor P_n is left-adjoint to the inclusion $\mathsf{Exc}^n \to \mathsf{Fun}$. To show this we have to demonstrate a natural isomorphism $\mathsf{Fun}(F, G) \cong \mathsf{Exc}^n(P_n F, G)$ for an arbitrary functor F and n-excisive functor G. The map $\mathsf{Exc}^n(P_n F, G) \to \mathsf{Fun}(F, G)$ is not so interesting: it is given by precomposition with $F \to P_n F$. The other half is more interesting: a map $F \to G$ induces a square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d] \\ P_n F \arrow[r] \arrow[ur, dotted] & P_n G. \arrow[u, dotted, bend right] \end{tikzcd}$$

The right-hand map is an equivalence since G is already n-excisive, and so we get a composite map $P_n F \to G$ by following the bottom edge and then the homotopy inverse to the right edge. One can check that these two maps are inverses, so give the desired adjunction.[3]
- Finally we have $P_n P_{n+k} \simeq P_n$, since the composite $P_n P_{n+k}$ also satisfies the same left adjoint property to the inclusion $\mathsf{Exc}^n \to \mathsf{Fun}$.

4. Simple examples

By assuming the existence of a zero object, the functors Σ and Ω are defined in great generality by pushing out against the two maps to the zero object and pulling back along the two maps from the zero object respectively. This construction coincides with the usual one in the categories of pointed spaces and of spectra. Expanding out the definitions of $\mathscr{X}$ and $T_1 F$, we see that when $F(1) = 1$ we have the formula $T_1 F(X) = \Omega F(\Sigma X)$. This allows us to compute two examples right off: taking F to be the identity functor on pointed spaces, we compute

$$P_1 \operatorname{id}_{\mathsf{Spaces}} = \operatorname{colim}_k T_1^k \operatorname{id}_{\mathsf{Spaces}} = \operatorname{colim}_k \Omega^k \operatorname{id}_{\mathsf{Spaces}} \Sigma^k = \Omega^\infty \Sigma^\infty,$$

sometimes called Q and of immense classical interest. Performing this same computation for spectra yields $P_1 \operatorname{id}_{\mathsf{Spectra}} = \operatorname{colim}_k \Omega^k \Sigma^k = \operatorname{colim}_k \operatorname{id}_{\mathsf{Spectra}} = \operatorname{id}_{\mathsf{Spectra}}$, meaning that the identity functor on spectra is 1-excisive. In turn, this means that it is k-excisive for all k, so that $P_k \operatorname{id}_{\mathsf{Spectra}} \simeq \operatorname{id}_{\mathsf{Spectra}}$.[4]

More complicated examples of similar flavor abound. For instance, Kuhn [1] claims that P_1 of the identity on augmented, commutative $\mathbb{S}$-algebras is $R \mapsto R \vee TAQ(R)$. This is follows from work of Basterra and Mandell [2], which draws on Basterra-McCarthy [3] and Schwede [4]. Generally, one can try to describe what "stabilization" means for some broad class of categories; Schwede [5] does this in different language in for simplicial algebraic theories.

The skeptical reader might complain that all these examples are first derivatives, and we ought to be talking about something of higher order to see some genuinely new examples. Unfortunately — but not surprisingly — it turns out to be difficult to compute any further examples with just the technology stated so far. Going back to the analogy with Taylor expansions of functions, we saw two definitions of $P_n f$: one that looked simple to restate in the language of homotopy functors and one that looked computationally useful — and that they were equivalent was a nontrivial fact. Something similar is going to happen now for us; we have successfully constructed a Taylor tower for any suitable functor F and object X:

$$F(X) \to \cdots \to P_3 F(X) \to P_2 F(X) \to P_1 F(X) \to P_0 F(X).$$

Thinking of these things as polynomial approximations of increasing top degree, the "difference" between the nth and $(n-1)$th levels should be exactly one term in the Taylor summation formula. So, let's define $D_n F$ by $D_n F = \operatorname{fib}(P_n F \to P_{n-1} F)$; we quickly see that the functor $D_n F$ is said to be n-homogeneous, meaning that it is n-excisive and is $(n-1)$-reduced, i.e., $P_{n-1} D_n F \simeq 1$. In future talks, we will be principally interested in studying these $D_n F$; for instance, we'll show that when F is a self-map of the category of spectra, we get a formula $(D_n F)(X) = (C_n \wedge X^{\wedge n})_{h\Sigma_n}$, which is eerily similar to the summand $\frac{f^{(n)}(a)(x-a)^n}{n!}$ in the Taylor formula.[5] Most importantly, we will find out that these D_n are much more readily computable than their cousins P_n.

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[3] Here, working in an ∞-categorical setting is to our honest advantage. The inverse $P_n G \to G$ cannot be reliably chosen so that everything strictly commutes, but instead P_n is a left adjoint in the sense of ∞-categories to i_n.

[4] This is an if-and-only-if: the map $\operatorname{id} \to T_1 \operatorname{id}$ is an equivalence exactly when the underlying category in question is a stable ∞-category.

[5] This gives new plausibility to one of the above examples: the identity on spectra $X \mapsto (\mathbb{S} \wedge X^{\wedge 1})_{h\Sigma_1}$ looks exactly like a degree 1 polynomial.

For a brief moment, let's posit this description of the functors D_n and compute something. Recall that because P_n is a left adjoint it commutes with colimits.[6] The seasoned homotopy theorist will recall the Snaith splitting

$$\Sigma^\infty\Omega^\infty\Sigma^\infty X \simeq \bigvee_{j=1}^{\infty}(\Sigma^\infty X^{\wedge j})_{h\Sigma_j}$$

for 0-connected spaces X. This looks awfully similar to what we've been discussing, and we can use formal properties of P_n, along with the fact that $X \mapsto X^{\wedge n}_{h\Sigma_n}$ is n-homogeneous, to compute P_n of this functor:

$$(P_n\Sigma^\infty\Omega^\infty\Sigma^\infty)(X) \simeq P_n\left(\bigvee_{j=1}^{\infty}(\Sigma^\infty(-)^{\wedge j})_{h\Sigma_j}\right)(X) \simeq \left(\bigvee_{j=1}^{\infty}P_n(\Sigma^\infty(-)^{\wedge j})_{h\Sigma_j}\right)(X) \simeq \bigvee_{j=1}^{n}(\Sigma^\infty X^{\wedge j})_{h\Sigma_j}.$$

In turn, $D_n\Sigma^\infty\Omega^\infty\Sigma^\infty$ is the difference between P_n and P_{n-1}:

$$(D_n\Sigma^\infty\Omega^\infty\Sigma^\infty)(X) \simeq (\mathbb{S}\wedge\Sigma^\infty X^{\wedge n})_{h\Sigma_n}.$$

As something to look forward to, once we study the functors D_n more carefully we'll be able to approach this problem from the other direction, concluding with the Snaith splitting.[7] [8]

5. Convergence

But we don't actually know these facts about D_n yet, so we'll have to occupy our time with something else. Luckily, this is easy: suppose again that D has a notion of homotopy groups. As algebraic topologists, now that we've drawn a tower of fibrations we should feel an overwhelming compulsion to investigate the associated Bousfield-Kan spectral sequence, with signature

$$E^1_{p,q} = \pi_p D_q F(X) \overset{cond}{\Rightarrow} \pi_p P_\infty F(X), \quad \text{with} \quad d^r_{p,q} : E^r_{p,q} \to E^r_{p-1,q+r}.$$

The first step in analyzing this spectral sequence is to compare the limit $P_\infty F(X) = \lim_k P_k F(X)$ with $F(X)$ itself; in the case that the natural map $F(X) \to P_\infty F(X)$ is an equivalence, the Taylor tower for F is said to converge at X. If $P_\infty F$ converges to F for all inputs, F is said to be entire.

In the case that the tower converges to F at X, we at least get conditional convergence. One way to ensure strong convergence is to force a vanishing line of positive slope into the spectral sequence. To this end, we make two definitions about the behavior of F with respect to connectivity:

- F satisfies property $E_n(c,\varkappa)$ when for any strongly co-Cartesian $(n+1)$-cube $\mathscr{X}$ with all 1-vertices $s \in S$ having the map $\mathscr{X}\emptyset \to \mathscr{X}\{s\}$ at least k_s-connected for $k_s \geq \varkappa$, then the map $F\mathscr{X}(\emptyset) \to \lim F|_{\mathscr{P}_0 S}$ is $(-c+\sum_s k_s)$-connected.
- Finally, F is said to be ρ-analytic if there exists a d so that F is $E_n(n\rho - d, \rho + d)$ for all $n \geq 1$.

If F is ρ-analytic and X is k-connected for some $k > \rho$, then the map $F(X) \to P_q F(X)$ is at least $(d+k+q(k-\rho))$-connected and hence $D_q F(X)$ is $(d+k+(q-1)(k-\rho))$-connected. Thus, the groups $E^1_{p,q}$ vanish when they satisfy the inequality

$$\begin{aligned} p &\leq d+k+(q-1)(k-\rho) \\ q &\geq \frac{p-d-k}{k-\rho}+1 = p\cdot\left(\frac{1}{k-\rho}\right) - \frac{d+\rho}{k-\rho}. \end{aligned}$$

This gives a vanishing line with positive slope, and hence a strongly convergent spectral sequence.[9] If D is Spectra and E is a connective spectrum, then there is a similar spectral sequence for $E_* P_\infty F(X)$ given by smashing through

[6] D_n is defined as the fiber of two functors that commute with sequential colimits and finite limits, so it does too by the standing assumption on D. If D is additionally stable, then fiber and cofiber sequences agree, and we can produce $D_n F$ as the cofiber of $\Omega P_{n+1}F \to \Omega P_n F$. Remarking that P_n commutes with general colimits and Ω does as well, as it's an autoequivalence of D, we see that D_n commutes with general colimits too.

[7] Thinking of $\mathbb{S}$ as the unit for the monoidal structure just like 1 is the unit for multiplication, this gives an amusing comparison between this functor and the exponential function.

[8] Amusingly, $X^{\wedge n}_{h\Sigma_n}$ is sometimes written $D_n X$, called the "nth extended power." This coincidence of notation is almost certainly accidental.

[9] In particular, this means that the Taylor tower converges to F at X. These properties additionally tell us how fast the convergence is.

the tower with E, since $E \wedge X$ is at least as connected as X. For an Eilenberg-Mac Lane spectrum HR, we also get a spectral sequence for $HR^* P_\infty F(X)$, where HR is an ordinary cohomology theory.

6. Existence of $\mathscr{Y}$

Let's quickly regurgitate Rezk's proof of the existence of the factorization of a Cartesian cube $\mathscr{Y}$, which works by constructing an n-cube of n-cubes. Let $\mathscr{X}$ be a strongly co-Cartesian n-cube in C, indexed by $\mathscr{P}S$ for a set S with $|S| = n$, and let $F : \mathsf{C} \to \mathsf{D}$ be as before. For any $T \subseteq S$, define $\mathscr{X}_T$ by

$$\mathscr{X}_T(R) = \operatorname{colim}\left(\mathscr{X}(R) \xleftarrow{\text{fold}} \coprod_{t \in T} \mathscr{X}(R) \xrightarrow{\text{cube map}} \coprod_{t \in T} \mathscr{X}(R \cup \{t\}) \right).$$

Picking $T = \emptyset$ causes the colimit to collapse, giving $\mathscr{X}_\emptyset = \mathscr{X}$. There is also a natural map $\alpha_T : \mathscr{X}_T \to \mathscr{X} * T$, using the definition

$$X * T = \operatorname{colim}\left(X \xleftarrow{\text{fold}} \coprod_{t \in T} X \to \coprod_{t \in T} 1 \right).$$

Putting these two facts together, we factor the map of cubes $(t_{n-1}F)(\mathscr{X}) : F\mathscr{X} \to (T_{n-1}F)\mathscr{X}$ as

$$F(\mathscr{X}(R)) = F(\mathscr{X}_\emptyset(R)) \xrightarrow{\text{univ. property}} \lim_{T \in \mathscr{P}_0 S} F(\mathscr{X}_T(R)) \xrightarrow{\alpha} \lim_{T \in \mathscr{P}_0 S} F(\mathscr{X}(R) * T) \simeq (T_{n-1}F)(\mathscr{X}(R)).$$

When $\mathscr{X}$ is strongly co-Cartesian, the diagram in the colimit defining $\mathscr{X}_T(R)$ picks out a corner of the cube $\mathscr{X}$, and so we get a natural weak equivalence $\mathscr{X}_T(R) \simeq \mathscr{X}(R \cup T)$. The maps $\mathscr{X}(R \cup T) \to \mathscr{X}(R \cup \{t\} \cup T)$ are isomorphisms when $t \in T$, and thus if T is nonempty then a punctured face not containing t of the punctured cube $F \circ \mathscr{X}_T|_{\mathscr{P}_0 S}$ is a duplicate of the face across t, and hence $F \circ \mathscr{X}_T$ is Cartesian. Therefore $\lim_{T \in \mathscr{P}_0 S} F(\mathscr{X}_T(R))$ is a homotopy limit of Cartesian cubes, and so is Cartesian itself, applying a lemma from the previous talk. We take $R \mapsto \lim_{T \in \mathscr{P}_0 S} F(\mathscr{X}_T(R))$ to be our $\mathscr{Y}$.

References

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