SOME FORMAL GEOMETRY AROUND K-THEORY

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ABSTRACT. Classical K–theory ranks among the most well-studied objects in homotopy theory, and its many rich structures enjoy interpretations which are simultaneously geometric and algebraic. The "chromatic" perspective on homotopy theory promotes the organization of such algebraic information through algebraic geometry and uses this to highlight useful patterns and generalizations. I'll explain how such techniques apply to K–theory, including a sketch of how they show the existence of the complex σ–orientation, and then speculate about how a computation joint with Hughes and Lau suggests the presence of an interesting new sequence of infinite loopspaces over $BU[2k, \infty)$.

1. FORMAL SCHEMES FOR SPACES

The main goal of this talk is to communicate a way to organize computational results from algebraic topology in your head. If you flip back through the literature in the 70s and 80s (and we will do some of that ourselves in a moment), you'll find yourself very envious of such a system. People back then were writing these enormous papers with enormous manipulations of enormous formulas, and there was a real industry built around having sufficient facility with, say, the formula for the right–unit for Brown–Peterson cohomology, or with being adept with multi-index bookkeeping. This was very hard work then, and it's fairly hard work now to go back and try to understand what these topologists were up to. Attempting to untangle any of it will imbue you with an immediate appreciation for any kind of method that will allow you to compress one of these results into a small space.

I don't know who first considered the following method, but I do know that the majority of its appearance in the literature is connected, directly or indirectly, to Neil Strickland. The essential idea is to directly apply algebraic geometry to the situation: rather than associating to a space X the cohomology ring E^*X, we go one step further and associate the scheme $\operatorname{Spec} E^*X$ (over $\operatorname{Spec} E^*$). There's a clear caveat here: algebraic geometry interprets commutative rings, so E^* and E^*X had better be commutative, and the easiest way to enforce this is by restricting attention to spaces with E^*X *even–concentrated*. Secondly, it turns out to be useful to remember some of the topological structure associated to the original space: the sorts of X we consider in homotopy theory are all "CW", and the "C" means that they're exhausted by their compact subspaces: $X = \operatorname{colim}_\alpha \{X_\alpha\}_\alpha$. The cohomology E^*X_α of any one of these individual spaces is a finite-dimensional E^*–algebra,[1] and so we form a *formal scheme* from the system

$$X_E := \operatorname{Spf} E^*X := \{\operatorname{Spec} E^*X_\alpha\}_\alpha.$$

The prototypical example of this construction is its value on $X = \mathbb{CP}^\infty$ for ordinary cohomology $E = H\mathbb{F}_p$. As X has a presentation as a cell complex, it's sufficient to take the subsystem of finite subcomplexes to define X_E. In this case, the finite subcomplexes are $X_n = \mathbb{CP}^n$, with cohomology $H\mathbb{F}_p^*\mathbb{CP}^n = \mathbb{F}_p[x]/x^{n+1}$, and so altogether

$$\mathbb{CP}^\infty_{H\mathbb{F}_p} = \widehat{\mathbb{A}}^1_{\mathbb{F}_p},$$

where $\widehat{\mathbb{A}}^1_R = \operatorname{Spf} R[\![x]\!]$ is the "formal affine line".[2] We can make two immediate further observations:

(1) The condition that a cohomology theory E admit an isomorphism $E^*\mathbb{CP}^\infty \cong E^*[\![x]\!]$ is called the *complex–orientability* of E. In our language, E being complex orientable exactly means that $\mathbb{CP}^\infty_E$ is (non-canonically) isomorphic to a formal affine line.

(2) The space $\mathbb{CP}^\infty = BU(1)$ has a map classifying the tensor of complex line bundles:

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\otimes} \mathbb{CP}^\infty.$$

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[1]Actually, some care is required here, since X_α need not all have even–concentrated cohomology even if X does. In the examples of interest, this won't be an issue — for instance, it suffices for H_*X to be even and torsion-free. I'd advise you to ignore the wrinkle for now.

[2]For that matter, a prototypical formal scheme comes from taking the germ of a point in a Noetherian scheme.

Just by checking degrees, one can calculate that the induced map

$$\mathbb{CP}^\infty_{H\mathbb{F}_p} \times \mathbb{CP}^\infty_{H\mathbb{F}_p} \to \mathbb{CP}^\infty_{H\mathbb{F}_p}$$

acts on points by $(x, y) \mapsto x + y$, and so a yet better name for $\mathbb{CP}^\infty_{H\mathbb{F}_p}$ is $\widehat{\mathbb{G}}_a$. In general, when E is complex–orientable $\mathbb{CP}^\infty_E$ carries the structure of a commutative 1–dimensional smooth formal group.[3]

2. A SECOND EXAMPLE: $BU(n)_E$

To a certain crowd, illustrating features of this functor as applied to $\mathbb{CP}^\infty$ is old hat; anytime complex orientations are mentioned, formal group laws also arise, and we really weren't exploring anything beyond that. The thesis I want to advance is that Neil's construction continues to be useful when applied to other spaces too, and for slightly more serious example in the same vein we'll explore the spaces $BU(n)$.

2.1. $BU(n)_E \cong \widehat{\mathbb{A}}^n$.

The space $BU(n)$ classifies complex vector bundles of rank n; suppose that we have such a bundle V over a space X. Associated to V we can form its fiberwise projectivization $\mathbb{P}(V)$, which is a $\mathbb{CP}^{n-1}$–bundle over X. The space $\mathbb{P}(V)$ itself comes equipped with a canonical line bundle, and hence a map

$$\mathbb{P}(V) \xrightarrow{p_V} X \times \mathbb{CP}^\infty.$$

Theorem. *When E is complex oriented and E^*X is even, the induced map E^*X–module map takes the form*

$$
\begin{array}{ccc}
E^*(X \times BU(1)) & \longrightarrow & E^*\mathbb{P}(V) \\
\| & & \| \\
E^*X \otimes E^*[\![x]\!] & \longrightarrow & E^*X \otimes E^*[\![x]\!]/\langle c_*(V),\ \text{of degree } n \rangle.
\end{array}
$$

Using this theorem, we define the *Chern classes of V* by

$$0 = x^n - c_1(V)x^{n-1} + c_2(V)x^{n-2} + \cdots + (-1)^n c_n(V).$$

This polynomial is called $c_*(V)$, the *total Chern class of V*; it is a monic polynomial generating the ideal corresponding to the quotient ring $E^*\mathbb{P}(V)$. A second basic theorem declares that these classes c_j account for all of $E^*BU(n)$:

Theorem. *A complex orientation of E begets an isomorphism $E^*BU(n) \cong E^*[\![c_1, \ldots, c_n]\!]$.*

In our language, this allows us to identify the formal scheme $BU(n)_E$ as the smooth formal scheme $BU(n)_E \cong \widehat{\mathbb{A}}^n$.

2.2. $BU(n)_E \cong \mathrm{Div}_n^+ \mathbb{CP}^\infty_E$.

We can do better than this. Applying our formal scheme functor to p_V, the same theorem asserts that $\mathbb{P}(V)_E \to X_E \times \mathbb{CP}^\infty_E$ is a closed inclusion, i.e., an effective divisor of degree n on $\mathbb{CP}^\infty_E$, or a E^*X–point of $\mathrm{Div}_n^+ \mathbb{CP}^\infty_E$.

Theorem. *A complex orientation of E begets an isomorphism $BU(n)_E \cong \mathrm{Div}_n^+ \mathbb{CP}^\infty_E$.*

Let's take the time to show that this is a serious description, carrying much more information that you might think. To begin, recall that iterated projectivization can be used to prove the following essential theorem:

Theorem (Splitting principle). *Suppose $V \downarrow X$ is a rank n complex vector bundle on X. There exists a natural space $f : Y \to X$ over X for which …*

    (1) *… the induced map $f^* : E^*X \to E^*Y$ is an injective map of rings.*
    (2) *… the pullback bundle f^*V has a canonical splitting into complex lines:*

$$f^*V \cong \bigoplus_{j=1}^n \mathcal{L}_j.$$

That is, the classifying map $X \to BU(n)$ lifts across the direct sum map

$$\overbrace{BU(1) \times \cdots \times BU(1)}^{n \text{ times}} \xrightarrow{\oplus} BU(n).$$

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[3]The reader is invited to check $\mathbb{CP}^\infty_{KU} \cong \widehat{\mathbb{G}}_m$.

Applying the splitting principle to V and using properties of the total Chern class c_*, we then have

$$c_*(f^*V) = x^n - f^*c_1(V)x^{n-1} + \cdots + (-1)^n f^*c_n(V) = c_*\left(\bigoplus_{j=1}^n \mathscr{L}_j\right) = \prod_{j=1}^n c_*(\mathscr{L}_j) = \prod_{j=1}^n (x - c_1(\mathscr{L}_j)).$$

These are called the "Chern roots" of $c(f^*V)$, and it's now plain that the splitting principle is a topological lift of the factorization of the Chern polynomial. The space Y enlarges the cohomology ring to be sufficiently solveable so that roots exist, and then additionally the roots are realized by complex lines. This digression is meant to provide some intuition about how the isomorphism $BU(n)_E \cong \mathrm{Div}_n^+ \mathbb{CP}_E^\infty$ behaves: the point corresponding to a vector bundle V is mapped to the divisor which, after sufficient base extension, is given by the formal sum of its Chern roots.

Additionally, the spaces $BU(n)$ come with formal sum and tensor product operations:

$$BU(n) \times BU(m) \xrightarrow{\oplus} BU(n+m), \qquad\qquad BU(n) \times BU(m) \xrightarrow{\otimes} BU(n \cdot m).$$

The first of these is easy to account for: the total Chern class has $c_*(V \oplus W) = c_*(V) \cdot c_*(W)$, so the induced map

$$
\begin{array}{ccc}
BU(n) \times BU(m) & \xrightarrow{\quad\oplus\quad} & BU(n+m) \\
\| & & \| \\
\mathrm{Div}_n^+ \mathbb{CP}_E^\infty \times \mathrm{Div}_m^+ \mathbb{CP}_E^\infty & \dashrightarrow{\;+\;} & \mathrm{Div}_{n+m}^+ \mathbb{CP}^\infty
\end{array}
$$

sends a pair of divisors to their formal sum. The tensor product is easiest to describe through the splitting principle:

$$c(V \otimes W) = c\left(\left(\bigoplus_{j=1}^n \mathscr{L}_j\right) \otimes \left(\bigoplus_{k=1}^m \mathscr{H}_k\right)\right) = \prod_{j,k} c(\mathscr{L}_j \otimes \mathscr{H}_k).$$

From the example at the top of the hour, we know what the Chern polynomial of a tensor product of lines corresponds to: we're using the group structure of $\mathbb{CP}_E^\infty$ to build the formal sum

$$\left(\sum_{j=1}^n [a_j]\right) \cdot \left(\sum_{k=1}^m [b_k]\right) = \sum_{j,k} [a_j + b_k].$$

Collectively, these isomorphisms efficiently describe a ring scheme structure on $\coprod_n \mathrm{Div}_n^+ \mathbb{CP}_E^\infty$ reflecting all of the structure on the cohomology rings $E^*BU(n)$.

3. $\underline{k}U_{2k}$

Given these descriptions, it's easy to take the colimit in n to get a description of BU_E: just as BU classifies stable vector bundles of virtual rank zero, $BU_E \cong \mathrm{Div}_0 \mathbb{CP}_E^\infty$ classifies stable divisors of virtual weight zero. Eliminating this weight condition, we also have $(BU \times \mathbb{Z})_E \cong \mathrm{Div}\, \mathbb{CP}_E^\infty$. These two spaces suggest a new avenue of generalization, as they are both spaces in the connective complex K–theory spectrum:

$$BU \times \mathbb{Z} \simeq \underline{k}U_0, \qquad\qquad BU \simeq \underline{k}U_2.$$

The next space in this sequence is also very accessible. It lies in a fiber sequence

$$
\begin{array}{ccc}
BSU & \longrightarrow & BU \xrightarrow{\;\det\;} BU(1) \\
\| & & \| \qquad\qquad \| \\
\underline{k}U_4 & \longrightarrow & \underline{k}U_2 \longrightarrow \mathbb{CP}^\infty.
\end{array}
$$

Does is map $\underline{k}U_4 \to \underline{k}U_2$ map of of inf loopspaces?

For complex–orientable E, the associated Serre spectral sequence is collapsing and we have an induced short exact sequence of group schemes

$$
\begin{array}{ccc}
BSU_E & \longrightarrow & BU_E \longrightarrow BU(1)_E \\
\| & & \| \qquad\qquad \| \\
\mathrm{SDiv}_0 \mathbb{CP}_E^\infty & \longrightarrow & \mathrm{Div}_0 \mathbb{CP}_E^\infty \xrightarrow{\;\sigma\;} \mathbb{CP}_E^\infty,
\end{array}
$$

where σ is the summation map and "SDiv" denotes "special divisors", i.e., those which sum to zero.

After this space, things get complicated quickly. The fiber sequence

$$K(\mathbb{Z},3) \to BU[6,\infty) \to BSU$$

has a somewhat accessible Serre spectral sequence, but the higher analogues do not. In his PhD thesis, Bill Singer completed this calculation for mod–p cohomology using carefully iterated Eilenberg–Moore spectral sequences:

Theorem (Bill Singer; Bob Stong). *Take $E = H\mathbb{F}_2$. There is an isomorphism*

$$H\mathbb{F}_2^*(BU[2k,\infty)) = \frac{H\mathbb{F}_2^*(BU)}{\mathbb{F}_2[\theta_{2i} \mid \sigma_2(i-1) < k-1]} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-3}],$$

where "$\mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-3}]$" denotes the smallest sub-Steenrod-Hopf-algebra of $H\mathbb{F}_2^*(K(\mathbb{Z},2k-3))$ containing $\mathrm{Sq}^3 \iota_{2k-3}$ and $\theta_{2i} \equiv c_i$ modulo decomposables.

This presentation does not suggest any geometric description. Instead, using as motivation the fact that "Div" constructs a sort of free group scheme, Ando, Hopkins, and Strickland went looking for interesting free constructions laying around. Taking powers of the natural map $(\mathcal{L} - 1) : BU(1) \to BU \simeq \underline{kU}_2$ gives an interesting map

$$BU(1)^{\times k} \xrightarrow{f_k} \underline{kU}_{2k} \simeq BU[2k,\infty).$$

Some properties of this map are evident: it is symmetric under permuting the domain, and restricting it to the basepoint of any of the factors collapses the map. There is an interesting third property, most easily visible by postcomposing to BU. There, the associated divisor (i.e., point in BU_E) takes the form $\langle a_1, \ldots, a_n \rangle := \prod_i ([a_i] - [0])$. We then compute:

$$
\begin{aligned}
\langle a_1, \ldots, a_{n+1} \rangle &= ([0]-[a_1])([0]-[a_2])([0]-[a_3])\langle a_4, \ldots, a_{n+1} \rangle \\
&= ([0]-[a_1])[a_2]([0]-[a_3])\langle a_4, \ldots, a_{n+1} \rangle + \langle a_1, a_3, \ldots, a_{n+1} \rangle \\
&= ([0]-[a_1])([a_2]-[a_2+a_3])\langle a_4, \ldots, a_{n+1} \rangle + \langle a_1, a_3, \ldots, a_{n+1} \rangle \\
&= \langle a_1, a_2, a_4, \ldots, a_{n+1} \rangle - \langle a_1, a_2+a_3, a_4, \ldots, a_{n+1} \rangle + \langle a_1, a_3, a_4, \ldots, a_{n+1} \rangle
\end{aligned}
$$
$$\Rightarrow \langle a_2, \ldots, a_{n+1} \rangle - \langle a_1 + a_2, a_3, \ldots, a_{n+1} \rangle = \langle a_1, a_2, a_4, \ldots, a_{n+1} \rangle - \langle a_1, a_2+a_3, a_4, \ldots, a_{n+1} \rangle,$$

a kind of cocycle condition. The most important step of this computation is the transition from the second to the third line: we used the fact that $\mathrm{Div}_0 \mathbb{CP}_E^\infty$ is an ideal for $\mathrm{Div}\,\mathbb{CP}_E^\infty$. This informs the following lucky guess:

Theorem (Ando, Hopkins, Strickland). *For even–periodic cohomology theories E and $k \leq 3$,[4] there is a diagram*

This is a hard theorem: not only does that map have to be checked to be an isomorphism, but the mere existence of the symmetric power scheme needs to be checked. It's also an incredible theorem: suppose that E is an elliptic cohomology theory, so that $\mathbb{CP}_E^\infty$ comes with a chosen isomorphism to the formal group $\widehat{C}$ of some elliptic curve C. The "theorem of the cube" in algebraic geometry applied to C furnishes us with a canonical point in $MU[6,\infty)_E$, i.e., a canonical multiplicative map $MU[6,\infty) \to E$. Morally, as TMF is the "universal elliptic cohomology theory", one can take a homotopy inverse limit over the various choices of C to get a map

$$MU[6,\infty) \xrightarrow{\sigma} \mathrm{TMF}.$$

This map indeed exists and is the complex–geometric version of "the σ–orientation" or "Witten's string genus". The construction of this canonical point in $MU[6,\infty)_E$ uses in an essential way the schematic description, and it's difficult to conceive of finding the homotopy theoretic instantiation of this map without employing this language.

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[4]At $k = 2$, this scheme is not obviously equivalent to the SDiv_0 description above. To explain: the map δ factors through $\ker \sigma = \mathrm{SDiv}_0\, \widehat{\mathbb{G}}$; we will define an inverse ϕ. Set $\phi_n(\underline{a})$ for a tuple $\underline{a} \in \widehat{\mathbb{G}}^n$ to be $\phi_n(\underline{a}) = \sum_{j=1}^n [\sigma(\underline{a}_{<j}), a_j]$. This turns out to be Σ_n–invariant, so one can write ϕ_∞. This map has $\phi_\infty(\underline{a}+\underline{b}) = \phi_\infty(\underline{a}) + \phi_\infty(\underline{b}) + [\sigma(\underline{a}), \sigma(\underline{b})]$, so for $\underline{a}$ and $\underline{b}$ in $\ker(\sigma)$ it is a homomorphism. This is the desired inverse.

You'll also notice that we didn't gain many new cases with this theorem: we already understood $\underline{k U}_{2k}$ for $k \le 2$, and the Ando–Hopkins–Strickland theorem applies to $k \le 3$. At $k = 4$, we can already see what's getting in the way: the odd–degree class $\mathrm{Sq}^7 \mathrm{Sq}^3 \iota_{2k-3}$ becomes nonzero for the first time when $k = 4$, and the connection to formal geometry collapses in the presence of odd–degree information. Nonetheless, the schemes C_k continue to exist, and one can investigate them in their own right.

Theorem (Hughes, Lau, P.). *For $E = H\mathbb{F}_2$, the Cartier–dual scheme $C^k = \mathbb{D}(C_k)$ has an explicit and efficient presentation which can be computed as far as out as one cares. (It isn't very pretty, though.)*

Formal geometry or not, the class f_k still exists, and it induces a map

$$\mathcal{O} C^k \xrightarrow{f_k'} (H\mathbb{F}_2)_* BU[2k, \infty).$$

Given our explicit presentation, we can attempt to analyze this map. Since the source is an even–concentrated Hopf algebra, its image in the target will also consist of even classes. However, Singer's calculation indicates that restricting to the subalgebra of even classes in the target is not sufficient to make f_k' an isomorphism. Instead, there appears to be one other item to take into account: the Steenrod algebra $\mathcal{A}_* = \mathcal{O} \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ naturally coacts on both sides.

Conjecture (Hughes, Lau, P.). *The map f_k' is $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$–equivariant. Restricting the target to the Steenrod–Hopf–subalgebra of even classes* which have even diagonals, *this map becomes an isomorphism.*

We've verified this computationally in thousands of bidegrees. I can't imagine it isn't true, but I don't have a proof. This modest conjecture naturally leads to a more seriously speculative question: is there an infinite loopspace X_{2k} over $\underline{k U}_{2k}$ realizing this factorization? I have no real feelings about this either way, but I do have a philosophical soapbox to stand on. The platform of this talk is basically that algebraic geometry can be used to capture a lot of what we do—and can even lead us to proofs of important ideas in homotopy theory, as with the σ–orientation. Faced with the fact that these two computations don't line up, we're forced to admit one of two things: either formal geometry isn't quite capturing the natural object of complex K–theory and the formal geometry needs to be augmented, or complex K–theory isn't quite capturing the natural algebraic geometry and the spectrum needs to be augmented.

I'm tempted to give the latter viewpoint a fair shake. Geometers seem a little confused about what, morally, comes after $BU[6, \infty)$ and $B\mathrm{String} = BO[8, \infty)$. The Thom spectra for the spaces that come after also don't really seem to fit as nicely into homotopy theory; it's known, for instance, that $MO[9, \infty)$ can't participate in a (suitably structured) orientation for the height 3 Morava E–theory. It sure would be interesting if there were some other candidate spaces X_{2k} with a tighter bond to algebraic geometry and so a better shot at achieving these goals.

Here are three immediate stray thoughts about these proposed spaces:

(1) The spaces X_{2k} cannot themselves assemble into a single infinite loopspace. A result from the 1970s of Adams and Priddy shows that any spectrum with $BU[2k, \infty)$ as its zeroth space must be a shift of kU. This is a neat paper; it works by "running the Adams spectral sequence backwards". Borrowing cues from it could turn up interesting results about, say, what the homotopy of X_{2k} must look like.

(2) Old work of Steve Wilson gives a description of all sufficiently nice H–spaces local to a prime: they are produces of spaces appearing as $BP\langle m \rangle_k$ in the Ω–spectrum for truncated Brown–Peterson theory. It would probably be instructive to understand the cohomologies of these spaces (a calculation due to Kathleen Sinkinson) and then to compare them with the ring of functions on C^k.

(3) Incredibly, there are tools around (due to Alexander Zabrodsky) to delete odd classes from H–space *while preserving their H-spaceness*. These kinds of techniques could be useful here, but I suspect they'll be too crude to yield the kind of interesting result we're looking for.