# Qualifying exam syllabus

Eric Peterson

General information

Date and time : November 23rd, 2011, at 10:00am.

Location: 891 Evans Hall.

- Ming Gu (committee chair)
- Martin Olsson
- Christos Papadimitriou (outside committee member, CS department)
- Constantin Teleman (adviser)

**Major topic: Algebraic Topology** (Constantin Teleman, geometry)

**Homotopy:** The fundamental group. Covering spaces and their automorphisms. Van Kampen. Classifying spaces. Higher homotopy groups. Eilenberg-Mac Lane spaces. Relative homotopy. Fibrations and the long exact sequence. The Freudenthal suspension theorem. Postnikov-Whitehead decomposition. Whitehead's theorem. Milnor's theorem. Homotopy limits and colimits of spaces. Fiber and cofiber sequences in the un/stable categories.

**Homology:** Axiomatic co/homology. Singular and simplicial co/homology. Cellular co/homology. Excision. Mayer-Vietoris. The Hurewicz isomorphism. Künneth and universal coefficient theorems. The stable category. Examples of extraordinary theories, e.g. $KU$. Brown representability. Bott periodicity. The Atiyah-Hirzebruch spectral sequence. Computations with the Serre spectral sequence. The Steenrod algebra and its dual. Cup and cap products. The James construction, and the statement of James' theorem. The statement of the Dold-Thom isomorphism. The Gysin sequence of a spherical bundle.

References: Allen Hatcher's *Algebraic Topology.* Robert Switzer's *Algebraic Topology: Homology and Homotopy.*

**Major topic: Homological Algebra** (Martin Olsson, algebra)

Category of chain complexes on an abelian category, its homotopy category, and its derived category. Cones, cylinders, and suspensions. The structure of a triangulated category. Derived functors. Injective and projective resolutions. The functors Ext, Tor, and $R\lim$ for categories of modules, and their properties. The relationship between the Ext functor in an abelian category and extensions. The ring structure on $\mathrm{Tor}^R(R/N, R/M)$. Spectral sequences associated to filtered complexes and bicomplexes. Hypercohomology. The model category structure on nonnegatively-graded chain complexes. Simplicial methods in homological algebra: the Dold-Kan correspondence for abelian groups; simplicial rings; and the André-Quillen co/homology of rings (definitions and computations).

References: Charles Weibel's *Homological Algebra.*

**Minor topic: Computational Complexity** (Christos Papadimitriou, applied mathematics)

Definitions and containments of $L$, $NL$, $P$, $NP$, $PH$, and $PSPACE$. Hierarchy theorems for $DTIME$, $NTIME$, and $DSPACE$. $NP$-completeness of SAT, 3-SAT, subset sum, vertex cover, and Hamiltonian cycle problems. $PSPACE$ completeness of TQBF and generalized geography games. Karp-Lipton. Savitch's theorem. Immerman-Szelepcsényi. Interactive proofs and $IP = PSPACE$.

References: Michael Sipser's *Introduction to the Theory of Computation.* Sanjeev Arora and Boaz Barak's *Computational Complexity: A Modern Approach.* Christos Papadimitrou's *Computational Complexity.*

What follows is a rough transcription of what happened during my qualifying exam. The speakers are labelled as:

- **MO**: Martin Olsson
- **CP**: Christos Papadimitriou
- **EP**: Eric Peterson
- **CT**: Constantin Teleman

A reader should note that this transcript may make me sound more prepared than I actually was. There were several long pauses and suggestions from the audience on what I should be doing next, and I was dangerously stumble-y during the computational complexity section. Some advice for students looking at this in preparation for their own quals would be to very deliberately bite off *less* than you can chew when writing your syllabus.

# 1 Algebraic Topology

**CT**: Compute the fundamental group of a closed, oriented surface.

**EP**: A closed oriented surface $\Sigma$ decomposes as a connected sum of tori, and we induct on the number of tori in the sum. In the genus 2 case, for instance, we have a pushout square of spaces with corners handle $\leftarrow$ circle $\rightarrow$ handle and handle $\rightarrow \Sigma \leftarrow$ handle. Van Kampen states that this translates to a pushout of fundamental groups. The fundamental groups of the two handles and the circle are known: they are $\mathbb{Z} * \mathbb{Z}$ and $\mathbb{Z}$ respectively. The pushout of these groups is an amalgamated free product:

$$\pi_1\Sigma = (\mathbb{Z} * \mathbb{Z}) * (\mathbb{Z} * \mathbb{Z})/(aba'b' \sim cdc'd').$$

**CT**: Do you really mean $cdc'd'$, or might it be $cd'c'd$? These generate different subgroups. Abstractly it doesn't matter, but let's draw these generators on the surface and see what we get.

**EP**: Start by drawing a gluing diagram for a torus and puncturing its center. Then, the loop that winds around the puncture point deforms out to the loop that runs around the edges. In the glued-together picture of the surface, this corresponds to taking the loop that winds around the disk removed for the connected sum, marking it in four points, and pulling each of these quartered segments along the torus to $a$, $b$, $a'$, and $b'$ respectively. [I did this in a way so that the obvious orientations on the gluing diagram and my surface didn't match, which caused some confusion.] Then, we rotate this figure around to get the other handle, and the same loops on this rotated surface look like so. The important thing, then, is that the loop around the excised holes on the two handles have opposite orientations, and so with the loops as I've labelled them here, we have $(\mathbb{Z} * \mathbb{Z}) * (\mathbb{Z} * \mathbb{Z})/(aba'b' \sim (cdc'd')')$.

**MO**: What happened to the basepoints in all of these calculations?

**EP**: Van Kampen's theorem is really a statement about fundamental groupoids: a pushout of spaces turns into a pushout of $\Pi_1$-groupoids. If we pick our pushout to be one of pointed spaces, so that the basepoint lives in both decomposition pieces and their intersection, we can trade for a pushout of fundamental *groups* rather than groupoids. Additionally, I've been drawing the gluing seams on the handles as loops that don't touch this basepoint; I can pick any path from the excision edge to the basepoint to get the isomorphism I want. This doesn't ruin the argument I just gave if I pick the same, rotated path on the other handle.

**CT**: Describe the Atiyah-Hirzebruch spectral sequence.

**EP**: A CW-complex is a space that admits a filtration by subspaces $\mathrm{sk}^n$, where $\mathrm{sk}^n$ is formed out of $\mathrm{sk}^{n-1}$ by attaching $n$-cells, and such that the original space is the colimit of this filtration. The filtration quotients (i.e., the homotopy cofibers of the inclusion of one filtration level into the next) are bouquets of spheres, and so when we apply a homology functor $E$ to the filtration we produce a filtration spectral sequence whose $E^1$-page is as described and whose $E^2$-page is the cellular homology of the space with coefficients in $E_*(\mathrm{pt})$.

**CT**: For a general pair of spaces $X$ and $Y$, describe a spectral sequence computing $\pi_* \operatorname{Maps}(X, Y)$.

**EP**: One thing we can do is filter $Y$ by a tower of fibrations — for example, the Postnikov-Whitehead tower gives us a tower

$$\cdots \to Y\langle n\rangle \to \cdots \to Y\langle 2\rangle \to Y\langle 1\rangle \to Y,$$

where each map occurs as the fiber of a map to an Eilenberg-Mac Lane space. A map $X \to Y\langle n\rangle$ lifts through $Y\langle n+1\rangle \to Y\langle n\rangle$ if and only if the composite $X \to Y\langle n\rangle \to K(\pi_n Y, n)$ vanishes. So, by general spectral sequence machinery, we get a spectral sequence with $E^1$-page given by

$$\operatorname{Maps}(\Sigma^p X, K(\pi_q Y, q)) = \operatorname{Maps}(X, \Omega^p K(\pi_q Y, q)) = \operatorname{Maps}(X, K(\pi_q Y, q-p)) = H^{q-p}(X; \pi_q Y).$$

**CT**: Can you compare these two spectral sequences?

**MO**: I see that of them arises from filtering the source, and the other from filtering the target; are those comparable?

**EP**: Putting the two together should give you a bicomplex, and so maybe stably, so that $\pi_*$ is both a homotopy- and a homology-functor to settle Martin's doubts, we get a pair of spectral sequences related by general bicomplex machinery. Would you like to work this out?

**CT**: No, thanks.

# 2 Homological Algebra

**MO**: Compute $\operatorname{Tor}_*^{\mathbb{Z}/p^2}(\mathbb{Z}/p, \mathbb{Z}/p)$. [He also asked for Ext, but we never came back to it.]

**EP**: We build a free (and so projective) resolution of the complex $0 \to \mathbb{Z}/p \to 0$ by taking

$$\cdots \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \to 0,$$

with the map $\mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p$ relating the two. Tensoring this resolution with $\mathbb{Z}/p$ over $\mathbb{Z}/p^2$ gives the complex

$$\cdots \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \to 0.$$

Everything is in the kernel of these differentials and nothing is in the image, so the Tor groups are $\mathbb{Z}/p$ in all nonnegative dimensions.

**MO**: Your syllabus says that this has a ring structure. Can you compute it in this example?

**EP**: The idea is that we can do better than resolve by a complex of free modules: we can resolve by a whole DGA. In fact, all the groups are going to be the same, and we just need to give names to the things in the resolution we've already constructed. We'll write 1 for the generator of $\mathbb{Z}/p$ in degree 0, and $b$ for the generator of $\mathbb{Z}/p$ in degree 1, so that $d(b) = 1$. We have another generator in degree 2 to name, but $b^2$ is no good, since $b \cdot b = -b \cdot b = 0$ by graded-commutativity, so we'll call it $a^{[1]}$ instead with $d(a^{[1]}) = b$. The generator in degree 3 is named: it is $a^{[1]}b$ with $d(a^{[1]}b) = b \cdot b + a^{[1]} \cdot 1 = a^{[1]}$. The name $(a^{[1]})^2$ is only maybe good for the generator in degree 4, since $d((a^{[1]})^2) = 2a^{[1]}$, which may vanish if $p = 2$. Even if it is a good name, we will run into this problem eventually, with $d((a^{[1]})^p) = p(a^{[1]})^(p-1) = 0$, at which point we'll introduce a new generator $a^{[p]}$. This process is exhaustive, and in the end the DGA wraps up into the tensor product $\Lambda_{\mathbb{Z}/p^2}[b] \otimes \Gamma_{\mathbb{Z}/p^2}[a]$ of an exterior factor and a divided-power factor, with $a^{[n]} \cdot a^{[m]} = \binom{n+m}{n} a^{[n+m]}$. Then, we tensor this DGA with Z/p to get $\operatorname{Tor}_*^{\mathbb{Z}/p^2}(\mathbb{Z}/p, \mathbb{Z}/p) = \Lambda_{\mathbb{Z}/p}[b] \otimes \Gamma_{\mathbb{Z}/p}[a]$.

**MO**: Where does this ring structure come from, what theorem are you quoting?

**EP**: The theorem that $\operatorname{Tor}^R(R/n, R/m)$ supports a ring structure basically says that this construction exists: $R/n$ has a surjective map $R \to R/n$, which lets us start the process, and then we continue from there using the recipe described above.

**MO**: Is it unique? What would happen if I used a different DGA?

**EP**: I actually don't know. I know that in Cartan-Eilenberg there's a section on how to compute this product without invoking the DGA construction, so presumably there is a stronger uniqueness statement. I don't know it, though.

**CT**: Produce an example of a tower $A_*$ with nontrivial $R^1 \lim(A_*)$.

**EP**: I don't have an example on hand, but we can try to work one out. Let's start by recalling an explicit construction of the $R\lim^1$ functor for abelian groups: given a tower of abelian groups $A_*$, we can form a map $\Delta : \prod_i A_i \to \prod_i A_i$ by the formula

$$\Delta(\ldots, a_n, a_{n-1}, \ldots, a_2, a_1) := (\ldots, a_n - a_{n+1}, a_{n-1} - a_n, \ldots, a_2 - a_3, a_1 - a_2),$$

where some elements are pushed down one level to live in the same group. Then, a short exact sequence of towers $0 \to A_* \to B_* \to C_* \to 0$ gives a short exact sequence of groups $0 \to \prod_i A_i \to \prod_i B_i \to \prod_i C_i \to 0$, which maps to itself by applying these $\Delta$ maps to each tower. The snake lemma gives a sequence

$$0 \to \ker \Delta_A \to \ker \Delta_B \to \ker \Delta_C \to \operatorname{coker} \Delta_A \to \operatorname{coker} \Delta_B \to \operatorname{coker} \Delta_C \to 0,$$

which is the homological sequence of lim and $\lim^1$ that we wanted. Hence, we want a tower $A_*$ with nontrivial $\operatorname{coker} \Delta_A$. Obviously onto maps are no good; that's covered by Mittag-Leffer. The zero map is also no good. Maybe a sequence of injective maps, then, is the right thing to do, like the tower $\cdots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}$. We can write out a sequence of arithmetic expressions and try to guess an element not in the image...

**CT**: That sounds hard to do; I think truncating this tower at any finite stage when trying to solve the arithmetic expressions will not show what's going on. How might this sequence appear in topology?

**EP**: The reason a topologist cares about $R\lim^1$ is that it appears in the Milnor sequence. For a directed system of spaces $X_a$, it is a theorem that homology plays nice with the colimit: $H_* \operatorname{colim}_a X_a = \operatorname{colim}_a H_* X_a$. This is not so for cohomology, where we instead have the short exact sequence

$$0 \to R\lim{}^1 H^{n-1} X_a \to H^n \operatorname{colim}_a X_a \to \lim{}_a H^n X_a \to 0.$$

Another way to compute this middle term is the universal coefficient sequence

$$0 \to \operatorname{Ext}^1(H_{n-1}X, \mathbb{Z}) \to H^n X \to \operatorname{Hom}(H_n X, \mathbb{Z}) \to 0.$$

Let's assume that we have a sequence of spaces $X_a$ whose homology is $\mathbb{Z}$, concentrated in dimension $n-1$, and with connecting maps inducing multiplication by $p$. We start calculating these groups:

- $\operatorname{Hom}(H_n X, \mathbb{Z}) = \operatorname{Hom}(0, \mathbb{Z})$ vanishes.
- $H_{n-1}X = H_{n-1} \operatorname{colim} X_a = \operatorname{colim}(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \cdots) = p^{-1}\mathbb{Z}$.
- To compute the Ext group, trying to resolve $p^{-1}\mathbb{Z}$ by projectives will land us right back where we started, so we replace $\mathbb{Z}$ by the complex of injectives $0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Then, $\operatorname{Hom}(p^{-1}\mathbb{Z}, \mathbb{Q}) = \mathbb{Q}$. There is an evaluation map $ev_1 : \operatorname{Hom}(p^{-1}\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ which acts by $ev_1(f) = f(1)$. A point in $\operatorname{Hom}(p^{-1}\mathbb{Z}, \mathbb{Q}/\mathbb{Z}$ is determined by where it sends the elements $1, p^{-1}, p^{-2}, \ldots$, subject to the single constraint that $p^n f(p^{-m}) = f(p^{n-m})$; using this, one checks that $\ker ev_1 = \mathbb{Z}_p^\wedge$. Finally, the map $c : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces a map from the exact sequence $0 \to \mathbb{Z} \to \operatorname{Hom}(p^{-1}\mathbb{Z}, \mathbb{Q}) \xrightarrow{coev_1} \mathbb{Q}/\mathbb{Z} \to 0$ to the exact sequence $0 \to \mathbb{Z}_p^\wedge \to \operatorname{Hom}(p^{-1}\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{ev_1} \mathbb{Q}/\mathbb{Z} \to 0$, which is inclusion on the left-most factor, $c_*$ on the middle, and equality on the right. Applying the snake lemma to this diagram yields the long exact sequence
$$0 \to 0 \to \ker c_* \to 0 \to \mathbb{Z}_p^\wedge/\mathbb{Z} \to \operatorname{coker} c_* \to 0 \to 0,$$
giving $\operatorname{Ext}^0(p^{-1}\mathbb{Z}, \mathbb{Z}) = \ker c_* = 0$ and $\operatorname{Ext}^1(p^{-1}\mathbb{Z}, \mathbb{Z}) = \operatorname{coker} c_* = \mathbb{Z}_p^\wedge/\mathbb{Z}$. [It turns out that $\operatorname{Hom}(p^{-1}\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}_p$, but this is not necessary to do the Ext calculation.]
- $\lim_a H^n X_a = \lim_a(\cdots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}) = \mathbb{Z}_p^\wedge$.

Assembling this, we have a short exact sequence

$$0 \to R\lim{}^1(\cdots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}) \to \mathbb{Z}_p^\wedge/\mathbb{Z} \to \mathbb{Z}_p^\wedge \to 0,$$

which forces nontriviality of the $R\lim^1$ term. Such a sequence of spaces is given by localizing the circle away from $p$.

# 3 Computational Complexity

**CP**: State the containments of the complexity classes listed on your syllabus.

**EP**: We have a tower $L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE$. $PH$ contains $NP$, but does not contain $PSPACE$.

**CP**: Insert $\Sigma_2$ into this tower.

**EP**: $\Sigma_2$ consists of the class of alternating Turing machines which are allowed polynomially many disjunctive branchings to start and then can continue using conjunctive branchings. It therefore contains $NP$, which consists of Turing machines which are allowed polynomially many disjunctive branchings and then no conjunctive branchings.

**CP**: State Karp-Lipton and Immerman-Szelepcsényi.

**EP**: Karp-Lipton states that if SAT can be solved by a boolean circuit whose size is polynomial in the size of the SAT problem, then the polynomial hierarchy collapses and $\Sigma_2 = PH$. Immerman-Szelepcsényi states that $NL = coNL$, which you show by demonstrating that $\overline{PATH}$ is in $NL$, then adapting to the general case.

**CP**: I see you have a probabilistic complexity class listed here: $IP$. Where does $IP$ fit into the tower?

**EP**: It is equal to $PSPACE$.

**CP**: Can you prove this?

**EP**: To show $PSPACE \subseteq IP$, we take the TQBF problem, which is PSPACE-complete, and produce an interactive proof protocol that solves it. The main idea of the proof is to convert a totally quantified boolean formula with an arithmetization: a number which vanishes if and only if the formula is unsatisfiable. To do this, we take our formula and replace all instances of the boolean variable $x_j$ with the polynomial indeterminate $y_j$, all instances of $\overline{x_j}$ with $(1 - y_j)$, $\wedge$ with $\cdot$, $\vee$ with $+$, $\forall x_j$ with $\prod_{x_j=0,1}$, and $\exists x_j$ with $\sum_{x_j=0,1}$. The verifier transmits this expression to the prover, who calculates the numeric value of the expression, who strips off the leading quantifier from the expression and calculates the resulting polynomial, and who transmits both of these pieces of information back to the verifier. The verifier evaluates the polynomial at 0 and at 1, either adds or subtracts the values as indicated by the removed quantifier, and checks that his answer matches the prover's. Then, the verifier selects either 0 or 1 (whichever makes the polynomial nonzero, or randomly if either works), makes the substitution into the boolean expression, and transmits that new expression back to the prover. The cycle repeats until all quantifiers are removed and a complete set of satisfying assignments is found. There are a lot of details omitted here; for instance, we have to make sure that these polynomials never get too large. Should we go through them?

**CP**: No, that's all right. What about the other direction, is there anything to say?

**EP**: Yes, there's a bit to say, but it's not so bad. The main idea is that once you're given a description of a machine in $IP$, you can in polynomial space compute what the odds are that it accepts some given input, by propagating values through a reachability tree.

**CP**: Good. Suppose we have a circuit which has n inputs and n outputs. State where in the tower the decision problem falls of whether this circuit computes the successor function mod $2^n$.

**EP**: We can test whether for any particular input we get the right output, but it is hard to do all the input simultaneously. That it's easy to check that the circuit fails to give the right function means that it belongs to $coNP$. Since each number up to $2^n$ takes up only $\log 2^n = n$ space, the problem is also in $PSPACE$.

**CP**: Now, do the same for the decision problem of whether the function represented by the circuit is onto.

**EP**: If the function represented by the circuit is onto, then it's in fact a bijection, and so the reachability graph, defined to have nodes length $n$ strings of digits and an edge between $n$ and $m$ if $C(n) = m$, is a disjoint union of cycles if it's indeed onto. If it's not, then there's some vertex of degree at least 3, and correspondingly two inputs $n_1$ and $n_2$ with $C(n_1) = C(n_2)$. Given two such numbers, it's quick to check that the circuit fails to be onto, and so this is again in $coNP$. We can also enumerate all such pairs of numbers, this time using $2n$ space instead of $n$ space, hence it is again in $PSPACE$.

**CP**: Do the same for the decision problem of whether sufficient iterations of the machine applied to the string of all 0s will yield the string of all 1s.

**EP**: This is an actual reachability problem, using the graph just described. Reachability is achievable in $NL$, but our graph has $2^n$ nodes, so this reduction tells us checking this can be done in $NPSPACE$. But, because $NPSPACE(f(n)) = PSPACE(f(n)^2)$ and polynomials are closed under squaring, $NPSPACE = PSPACE$ and this decision problem is a $PSPACE$ problem.

**CP**: Consider the class $NTIME(n^2)$. How are these related to the complexity classes P and NP?

**EP**: Certainly $NTIME(n^2)$ is a subset of $NP$. It is not known whether $P \subseteq NTIME(n^2)$ or the other way around. Either is possible, in fact; if $P = NP$, for example, then $NTIME(n^2) \subseteq P$. The question is a little more subtle than this, though, because of the hierarchy theorems for $NTIME$.

**CP**: OK, so is it possible that $NTIME(n^2) = P$?

**EP**: It is not. $NTIME(n^2)$ is a proper subset of $NP$, because, for instance, $NTIME(n^2) \subsetneq NTIME(n^4)$. However, polynomials are closed under composition, and hence $P$ is closed under polynomial-time reductions. This is not true of $NTIME(n^2)$, and so they cannot be equal.